

Generating Families and Legendrian Contact Homology in the Standard Contact Space

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Abstract

We show that if a Legendrian knot in standard contact \mathbb{R}^3 possesses a generating family then there exists an augmentation of the Chekanov-Eliashberg DGA so that the associated linearized contact homology is isomorphic to singular homology groups arising from the generating family. We discuss the relationship between normal rulings, augmentations, and generating families. In particular, we provide an explicit construction of a generating family for a front diagram with graded normal ruling and give a method for computing linearized contact homology groups using the combinatorics of a normal ruling.

1 Introduction

Our main objects of study are Legendrian knots in the ‘standard contact space’, $(\mathbb{R}^3, \ker(dz - ydx))$, and their invariants of Legendrian isotopy. Aside from the underlying topological knot type there are two “classical” integer valued invariants known as the Thurston-Bennequin number and the rotation (or Maslov) number. For some time it was not known whether there could exist distinct Legendrian knot types in the standard contact space with the

*The second author received support from NSF VIGRE Grant No. DMS-0135345 during portions of this work

same classical invariants (including topological knot type). This question was answered in the affirmative independently by Chekanov and Eliashberg. Motivated by Floer theory, they developed a rich algebraic invariant which takes the form of a DGA (Differential Graded Algebra) generated by self intersections of the knot's Lagrangian projection. The differential is defined by counting immersed (or holomorphic) disks. Generating families have provided a second source of non-classical invariants in the work of Traynor and her collaborators [Tr] [NgTr] [JTr] and Chekanov-Pushkar [ChP]. The main results of this paper give an interpretation of a linearized version of the Chekanov and Eliashberg invariant in terms of generating families. This strengthens previously known links between the two classes of invariants [F] [FI] [K2] [NgTr] [S2] [JTr].

Given a 1-parameter family of functions $F : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$, $f_t = F(\cdot, t)$ the fiber-wise critical set is immersed (under a transversality assumption) into the standard contact space as a Legendrian submanifold, ℓ , and F is called a *generating family* for ℓ . Traynor [Tr] introduced a homological invariant obtained from generating families for a class of 2-component links in the solid torus and (with Jordan [JTr]) in \mathbb{R}^3 . A variation¹ of the Traynor-Jordan invariant which applies without additional assumptions on the knot type is the following:

Given a generating family for ℓ consider the *difference function*

$$w : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}, \quad w(x, y, t) = F(x, t) - F(y, t).$$

Take δ close enough to 0 so that the interval $(0, \delta)$ consists entirely of regular values of w and consider the shifted homology groups $\mathcal{G}H_*(F) = H_{*+n+1}(w \geq \delta, w = \delta; \mathbb{Z}_2)$. While the homology may depend on the choice of generating family, the set of graded groups $\{\mathcal{G}H_*(F)\}$ where F is any generating family (with restrictions on its behavior outside of a compact set) for ℓ forms a Legendrian isotopy invariant (See section 5 for a sketch of the proof).

A central result of this paper is

Theorem 5.3 *If F is a generic (as described in section 5) and linear at infinity generating family for ℓ then there exists a graded augmentation ε for the Chekanov-Eliashberg DGA of ℓ such that $\mathcal{G}H_*(F) \cong H_*^\varepsilon(\ell)$.*

¹The authors became aware of this version of the generating family invariant through a letter from Peter Pushkar to the first author [P]. He considers the homology groups as a possibility for defining a Legendrian homology invariant, but as far as we know has never published anything in this regard.

The *augmentation* ε is an algebra homomorphism of the Chekanov-Eliashberg DGA into \mathbb{Z}_2 which allows us to form a finite dimensional complex with homology groups $H_*^\varepsilon(\ell)$.

A type of decomposition of a knot's front diagram known as a *normal ruling* provides a central motivation. This notion arose independently in the work of the first author on existence of augmentations [F] and as a combinatorial invariant defined by Chekanov-Pushkar [ChP] motivated by generating families. Rulings provide a link between the Chekanov-Eliashberg invariant and generating families by combining the following two theorems:

Theorem 2.7 ([F] [FI] [S2]) *The Chekanov-Eliashberg DGA of ℓ admits a (ρ -graded) augmentation if and only if the front diagram of ℓ admits a (ρ -graded) normal ruling.*

Theorem 2.4 ([ChP]) *A Legendrian knot ℓ has a generating family if and only if the front diagram of ℓ has a graded normal ruling.*

Considering the two theorems in conjunction we see that it is possible to form integer graded linearized homology groups $H_*^\varepsilon(\ell)$ if and only if ℓ has a generating family. This suggests a generating family interpretation for the groups $H_*^\varepsilon(\ell)$ which is the aim of Theorem 5.3. It should be noted that a close variant of Theorem 5.3. is proved in [NgTr] and [JTr] for certain classes of 2-component links where the invariants are explicitly calculated.

Overview of the rest of the paper: In section 2 we recall necessary notions from Legendrian knot theory. The Chekanov-Eliashberg DGA is defined as well as augmentations, normal rulings, and generating families. In conclusion, we give the construction of a normal ruling from a generating family.

Section 3 gives a proof of the reverse implication in Theorem 2.4 via an explicit construction of a generating family for a Legendrian link admitting a normal ruling. Although, this result is stated in [ChP] as far as the authors know a proof has yet to appear in print.

In section 4, a version of the ‘splash’ construction is presented which we use for computations with the Chekanov-Eliashberg DGA. We give a proof of Theorem 2.7 in which the similarity between the construction of a normal ruling from either a generating family or an augmentation is emphasized. In

section 4.3 we present an alternate complex for computing linearized homology groups based on the combinatorics of a normal ruling without involving the usual counts of immersed disks. As an application we give a calculation of a particular linearized homology group for Legendrian knots arising from Ng's Mondrean diagram construction.

The statement and proof of Theorem 5.3 occupies all of section 5. The analogy between the proofs of Theorem 2.4 and 2.7 motivates the construction of an augmentation directly from a generating family used here. To compute the generating family homology groups we use a fiber-wise version of the Morse complex based on bifurcation data in a 1-parameter family of functions.

Finally, in section 6 we provide a simplified alternate proof of a weaker version of Theorem 5.3 (Theorem 6.1) which applies to the explicitly described generating families from section 3. It should be noted that the augmentation used in the proof of Theorem 6.2 is canonically chosen based on the underlying normal ruling while the augmentation used in Theorem 5.3 is constructed more directly from the generating family. Sabloff has proved a duality result for the linearized homology groups $H_*^\varepsilon(\ell)$ from which a corresponding result follows for the groups $\mathcal{GH}_*(F)$ according to Theorem 5.3. In Theorem 6.3 we include a proof of Sabloff duality for the groups $\mathcal{GH}_*(F)$ purely from the generating family perspective.

2 Survey of known results

2.1 Legendrian curves and their projections

A *Legendrian curve* in \mathbb{R}^3 (with respect to the standard contact structure) is an immersed smooth curve $x = x(t), y = y(t), z = z(t)$ satisfying the equation $y\dot{x} - \dot{z} = 0$ (that is, tangent to the distribution $\mathcal{C} = \{y dx - dz = 0\}$). The words Legendrian knots, Legendrian links, and Legendrian isotopy have obvious sense. There are two convenient projections of Legendrian curves. An xz projection, or a *front projection* of a generic Legendrian curve is a smooth curve with finitely many cusps but without vertical tangents. A front projection uniquely determines the Legendrian curve: the missing y coordinate is reconstructed as the slope of the tangent line. A front projection of a Legendrian knot may have self-intersections but no self-tangencies (the latters would correspond to self-intersections of the Legendrian curve in

space). An xy projection of a Legendrian curve is smooth; it determines the curve up to a translation in the direction of the z axis: the missing z coordinate is reconstructed as $\int y dx$. An xy projection of a closed Legendrian curve, in particular, of a Legendrian knot, is a self-intersecting smooth curve enclosing a zero area.

2.2 Classical invariants

There are two classical integral-valued Legendrian isotopy invariants of Legendrian knots (and links). The *Thurston-Bennequin number* $TB(\ell)$ of a Legendrian knot ℓ is the linking number $\text{lk}(\ell, \ell^+)$ where ℓ^+ is a curve obtained from ℓ by a small shift in the direction of a normal within the distribution \mathcal{C} . (For knots, this number does not depend on the direction of the normal.) The *rotation number* $R(\ell)$ of an *oriented* Legendrian knot ℓ may be defined as the rotation number of its xy projection.

Both $TB(\ell)$ and $R(\ell)$ have simple description in terms of a front diagram. In this article, we will need this only for $R(\ell)$. Let L be a front diagram of an oriented Legendrian knot ℓ . Cusps break L into non-self-intersecting parts, “strands”. Take one of the strands and attach to it some integer, k . Let us move from the chosen strand in the direction of the chosen orientation of L . Passing through a cusp to a new strand, we add 1 to our integer, if the new strand is above the old one and subtract 1 otherwise. When we return to the initial strand, our integer becomes some k' ; it is easy to see that $k' - k = 2R(\ell)$. It is important that if $R(\ell) = 0$, then every strand acquires a number (which we will call the *index*), and of two strands forming a cusp, the index of the upper one is one more than the index of the lower one. These indices are defined up to simultaneous adding the same integer to all of them. If $R(\ell) \neq 0$, then indices with similar properties are defined as residues modulo $2|R(\ell)|$.

2.3 The Chekanov–Eliashberg DGA

Consider a generic xy -diagram Γ of a Legendrian knot ℓ . Let S be the set of all crossings of Γ , and let $\mathbf{A} = \mathbf{A}(\Gamma)$ be a free associative unital \mathbb{Z}_2 -algebra generated by S . At every crossing $s \in S$, the diagram forms four corners, of which we declare two positive and two negative: if you approach the crossing s along the upper strand (that is, the strand with a bigger value of z), then

the corner to the right of you is positive and the corner to the left of you is negative.

For every $n \geq 0$ fix a convex planar domain P_n bounded by a piecewise smooth curve with $n+1$ corners numerated counterclockwise as v_0, v_1, \dots, v_n . For a crossing s , consider the set $I_n(s)$ of regular isotopy classes of orientation preserving immersions $f: P_n \rightarrow \mathbb{R}^2$ such that (1) $f(\partial P_n) \subset \Gamma$, (2) $f(v_0) = s$, (3) a neighborhood of v_0 covers a positive corner at s , and (4) for $i = 1, \dots, n$, a neighborhood of v_i covers a negative corner at $f(v_i)$. Put $I(s) = \cup_n I_n(s)$. The differential $d: \mathbf{A} \rightarrow \mathbf{A}$ is defined as a derivation such that $d(s) = \sum_{[f] \in I(s)} f(v_1) \dots f(v_n)$. It is proved in [Ch] that $d^2 = 0$.

There exists a natural grading of \mathbf{A} which assigns to each crossing s a degree which is an integer, if $R(\ell) = 0$, and a residue modulo $2|R(\ell)|$ otherwise. Here is the definition. Let us agree to count the measure of a positive corner as zero and the measure of a negative corner as 180° . If we leave s along the upper strand, then the number of revolutions until the first return to s is a half-integer. Double it. This is the degree of s ; modulo $2|R(\ell)|$ it does not depend on the choice of a direction on the upper strand.) With respect to this grading the differential d has the degree -1 .

The following result is the main achievement of the Chekanov–Eliashberg theory ([Ch],[El]).

Theorem 2.1 (Chekanov, Eliashberg) *The (graded) homology of $\mathbf{A}(\Gamma)$ is a Legendrian isotopy invariant of ℓ .*

Chekanov’s paper contains the following, more precise statement.

Theorem 2.2 (Chekanov) *The stable isomorphism type of $\mathbf{A}(\Gamma)$ is a Legendrian isotopy invariant of ℓ .*

Let us provide an explanation. A *stabilization* of $\mathbf{A}(\Gamma)$ is obtained from $\mathbf{A}(\Gamma)$ by adding two generators a, b with $\deg(a) = \deg(b) + 1$ and extending the differential by the formulas $d(a) = b$, $d(b) = 0$. The algebras $\mathbf{A}(\Gamma)$, $\mathbf{A}(\Gamma')$ are stably isomorphic if some of their iterated stabilizations are isomorphic. (It is clear that a stabilization does not affect the homology, so Theorem 2.1 follows from Theorem 2.2.)

In conclusion, let us notice that a Chekanov–Eliashberg DGA of a Legendrian knot may be constructed from a front diagram. This was observed by L. Ng [Ng3] who showed that xy - and xz -diagrams of the same knot, however different they may seem, may be made looking almost the same.

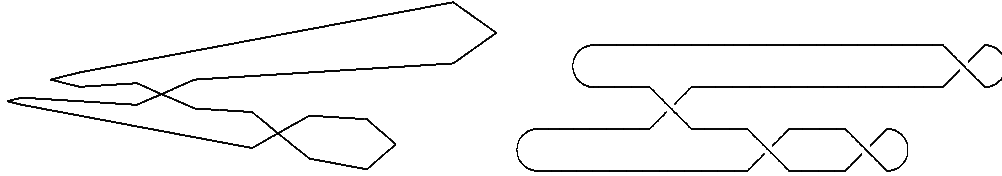


Figure 1: Ng's construction

A planar isotopy of the front diagram arranges the following. Away from crossings and cusps the arcs of the front diagram are straight lines such that the slope of these lines decreases as we move from the top arc towards the bottom arc. This will appear as several parallel horizontal lines on the xy -diagram. At a left cusp the newly appearing arcs should have their slopes fit this criterion. When we come to a crossing the slopes of the two adjacent arcs on the front diagram interchange causing an immediate crossing to appear on the xy -diagram and a crossing on the xz -diagram when the two lines eventually meet. Before a right cusp the slopes of the two arcs that will meet are interchanged. This produces an additional crossing on the xy -diagram which does not appear on the front diagram.

A diagram of this shape is shown in Figure 1, left. The corresponding xy diagram (Figure 1, right) looks, at least topologically, almost the same: crossings remain crossings, left cusps become roundings, right cusps become roundings preceded by additional crossings.

For this diagram, Γ , the algebra $\mathbf{A}(\Gamma)$ can be reconstructed from the initial front diagram, L , and we will denote it as $\mathbf{A}(L)$. The generators of $\mathbf{A}(L)$ correspond to crossings and right cusps of L . The degree of a generator corresponding to a crossing of strands S, S' where S has slope bigger than S' , is $\text{ind}(S) - \text{ind}(S')$ (ind stands for the index, see 2.2); the degree of a generator corresponding to a right cusp is $+1$.

2.4 Rulings

The following notion and its relationship on one hand with generating families and on the other with the Chekanov-Eliashberg DGA is a central motivation for this work. It was introduced in 2000 independently by Chekanov and Pushkar [ChP] and the first author [F].

Let L be a front diagram of a Legendrian knot ℓ . A *ruling* of L consists of (1) a correspondence between left and right cusps, and (2) for every pair

of corresponding left and right cusps, ℓ and r , two disjoint (except ℓ and r) paths within the diagram, with strictly increasing x -coordinate, from ℓ to r such that paths joining different pairs of cusps can meet only at crossings.

Obviously, the paths of a ruling never pass through the cusps, except the endpoints, and cover the whole diagram; this covering is one-fold, except the crossings and the cusps, where it is two-fold. In particular, any crossing belongs to two paths which may exchange or not exchange the strands passing through the crossing. In the first case the crossing is called a *switch*. A ruling is fully determined by the set of switches, so we can consider rulings as subsets of the set of crossings.

The notion of a ruling turns out to be useful only in the presence of the following “*normality condition*”. Assume that no two crossings of the diagram have the same x -coordinate. Let s be a switch, let p_u and p_ℓ be the upper and lower paths of the ruling passing through s , and let q_u and q_ℓ be the other paths joining the same cusps as p_u and p_ℓ . Let z, z_u , and z_ℓ be the z -coordinates of s and of the intersection points of q_u and q_ℓ with the vertical line through s . We call the switch *normal*, if $z_u > z > z_\ell$, or $z_\ell > z_u > z$, or $z > z_\ell > z_u$. (In the remaining cases, $z_u > z_\ell > z$, $z_\ell > z > z_u$, and $z > z_u > z_\ell$, the switch is *abnormal*.) A ruling is called normal if all the switches are normal.

Example. Let L be the front diagram in Figure 2 (this is a “standard” trefoil knot), with crossings s_1, s_2, s_3 . Then there are four rulings, $\{s_1, s_2, s_3\}, \{s_1\}, \{s_2\}, \{s_3\}$, and only one of them, $\{s_2\}$, is not normal.

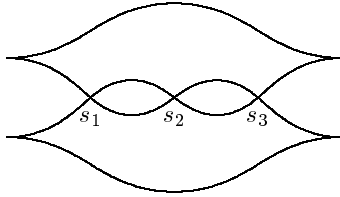


Figure 2: Front diagram of a standard trefoil

In the case when $R(L) = 0$, a ruling is called *graded*, if all the switches have degree 0. For example, all rulings of the diagram in Figure 2 are graded. In general case, a ruling of L is called ρ -graded, where ρ is a divisor of $2R(L)$, if the degrees of all switches are divisible by ρ . It is proved in [S2] that if $R(L) \neq 0$, then there never exists a normal 2-graded ruling.

Not every front diagram has a ruling (see Figure 3). But it is known

[ChP] that not only the existence of a normal ruling, but even the number of different normal rulings, as well as the number of different graded, or ρ -graded, normal rulings, is a Legendrian isotopy invariant. Moreover, a generic Legendrian isotopy between two front diagrams gives rise to a canonical bijection between the sets of normal rulings of these two diagrams. For further statements of this kind, see section 2.7.

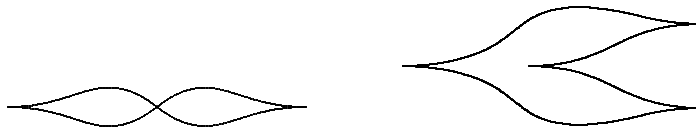


Figure 3: Front diagrams with no rulings

There are at least three different theorems stating that a certain property of a Legendrian knot is equivalent to the existence of a (in some cases, graded) normal ruling. We will observe these results in the three subsections below. These theorems show, in particular, that some, visibly unrelated, properties of front diagrams are mutually equivalent. Below, we will establish a relation between two of them.

2.5 Normal rulings and generating families of functions

Let $f_t(x)$, $t \in \mathbb{R}$, $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ be a generic family of functions such that, for some C , $f_t(x) = x_n$, if $|t| > C$ or $|x| > C$. Here “generic” means that except finitely many values of t (which we will call exceptional), f_t is a Morse function with all critical values being different, and each exceptional value of t correspond to one of the following events: a generic birth or death of a pair of critical points of adjacent indices; a generic collision of two critical values.

For a generic family $\{f_t\}$ consider the set

$$L = \{(t, z) \in \mathbb{R}^2 \mid z \text{ is a critical value of } f_t\}.$$

Obviously, L is a front diagram. We will say that $\{f_t\}$ is a *generating family* for L . Which front diagrams possess generating families? One restriction is obvious.

Proposition 2.3 *If L possesses a generating family, then $R(L) = 0$.*

Proof For any strand $S = \{(t, s(t)), t' \leq t \leq t''\}$ of L , there exists a continuous family $\{x_t \in \mathbb{R}^n, t' \leq t \leq t''\}$ where x_t is a critical point of f_t and $f_t(x_t) = s(t)$; obviously, the index of the critical point x_t is the same for all t , and we denote this index by $\text{ind}(S)$. At cusps of L , these indices behave as prescribed in Subsection 2.2; thus $R(L) = 0$.

Theorem 2.4 (Chekanov, Pushkar) *A front diagram L with $R(L) = 0$ possesses a generating family, if and only if it possesses a graded normal ruling.*

The *only if* part of this theorem is proved in [ChP]. The construction of a normal ruling for a front diagram with a generating family of functions is as follows. Let $f: \mathbb{R}^n \rightarrow \mathbb{R}$ be a Morse function with all critical values different and with $f(x) = x_n$ for $|x| > C$. For a real c , set $X_c = \{x \in \mathbb{R}^n \mid f(x) \leq c\}$. Call critical values $c_1 > c_2$ of f related, if, for a small $\varepsilon > 0$,

$$\begin{aligned} \dim_{\mathbb{Z}_2} H_*(X_{c_1+\varepsilon}, X_{c_2-\varepsilon}; \mathbb{Z}_2) + 1 &= \dim_{\mathbb{Z}_2} H_*(X_{c_1-\varepsilon}, X_{c_2+\varepsilon}; \mathbb{Z}_2) + 1 \\ &= \dim_{\mathbb{Z}_2} H_*(X_{c_1+\varepsilon}, X_{c_2+\varepsilon}; \mathbb{Z}_2) = \dim_{\mathbb{Z}_2} H_*(X_{c_1-\varepsilon}, X_{c_2-\varepsilon}; \mathbb{Z}_2) \end{aligned}$$

It turns out that the pairs of related critical values are disjoint; moreover, the whole set of critical values falls into the union of disjoint related pairs; moreover, the involutions arising in the sets of critical values of functions of generating family compose a graded normal ruling of the front diagram. (In particular, related critical values have adjacent indices.)

A different way to describe this ruling arises from the following proposition (which is used in topology since J.H.C.Whitehead's works of the 1930's). Note that this proposition will be important for us in the subsequent parts of this article.

Proposition 2.5 *Let V be a vector space over some field with a fixed ordered basis e_1, \dots, e_m , and let $d: V \rightarrow V$ be a linear transformation with the following two properties: (1) d is triangular, in the sense that $d(e_i) = \sum_{j>i} a_{ij}e_j$; (2) d is exact in the sense that $\text{Ker } d = \text{Im } d^2$. Then there exists a fixed point free involution $\tau: \{1, \dots, m\} \rightarrow \{1, \dots, m\}$ and a triangular basis change $e'_i = e_i + \sum_{j>i} b_{ij}e_j$ such that $d(e'_i) = e'_{\tau(i)}$, if $\tau(i) > i$, and $d(e'_i) = 0$, if $\tau(i) < i$. Moreover, the involution τ with this properties is uniquely determined by d .*

²There is a version of this proposition for the more general case $d^2 = 0$, but we do not need it.

For a non-exceptional value of t , consider the \mathbb{Z}_2 Morse complex associated with the function f_t and some Riemannian metric on \mathbb{R}^n (compatible with f_t in the sense of Morse-Smale). The total space V of this complex has a natural basis corresponding to the critical values of f_t ordered accordingly to the decreasing order in the set of critical values, and the differential d is triangular (obviously) and exact (because \mathbb{R}^n is acyclic, as well as the set $\{f_t < -C\}$). According to Proposition 2.5, this provides an involution in the set of critical values, and it can be checked that it is the same involution as before (in particular, it does not depend on the Riemannian metric).

The *if* part of Theorem 2.4 was proved by Pushkar, but, as far as we know, the proof has never been published. We provide a proof in Section 3. Our proof contains an explicit construction of a generating family for a front diagram with a graded normal ruling; this construction will be used again in section 6. Notice that the result claimed by Pushkar contains also a stable uniqueness statement for a generating family corresponding to a graded normal ruling.

2.6 Normal rulings and augmentations

An augmentation of the Chekanov-Eliashberg DGA, $A = A(\Gamma)$, is a unital ring homomorphism $\varepsilon: A(\Gamma) \rightarrow \mathbb{Z}_2$ such that $\varepsilon \circ d = 0$. An augmentation is completely determined by its restriction to the set of generators of A , that is, the set S of crossings of the xy -diagram Γ (the set of crossings and right cusps of the front diagram L for $A = A(L)$). An augmentation is called graded (ρ -graded), if $\varepsilon(s) \neq 0$, $s \in S$, implies $\deg s = 0$ ($\deg s \equiv 0 \pmod{\rho}$).

For an augmentation ε , set $A^\varepsilon = \text{Ker } \varepsilon / (\text{Ker } \varepsilon)^2$. This is a vector space with the basis $\{s^\varepsilon = s + \varepsilon(s) \mid s \in S\}$. If $a \in \text{Ker } \varepsilon$, then $da \in \text{Ker } \varepsilon$ (actually, $da \in \text{Ker } \varepsilon$ for any $a \in A$), and if $a \in (\text{Ker } \varepsilon)^2$, then $da \in (\text{Ker } \varepsilon)^2$ (if $a = bc$, $b, c \in \text{Ker } \varepsilon$, then $da = (db)c + b(dc) \in (\text{Ker } \varepsilon)^2$). Hence, $d: A \rightarrow A$ gives rise to a homomorphism $d^\varepsilon: A^\varepsilon \rightarrow A^\varepsilon$, and $(d^\varepsilon)^2 = 0$. If the augmentation ε is graded (ρ -graded), then A^ε is graded (ρ -graded), and d^ε has the degree -1 . Hence, there arises a homology H^ε which is graded (ρ -graded), if so is ε .

Theorem 2.6 (Chekanov [Ch]) *The set of all (all graded, all ρ -graded) homologies H^ε corresponding to all (all graded, all ρ -graded) augmentations of A is a Legendrian isotopy invariant.*

The problem of existence of an augmentation is solved by the following result.

Theorem 2.7 (Fuchs, Ishkhanov, Sabloff) *The algebra $A(\Gamma)$ possesses a (graded, ρ -graded) augmentation if and only if the corresponding front diagram possesses a (graded, ρ -graded) normal ruling.*

The *if* part of this theorem was proved by the first author in [F] (and this was one of the initial motivations for the notion of a normal ruling). The *only if* part was proved in [FI] and, independently, in [S2]. We will discuss the proofs in Section 4, where we will present, in particular, a direct (bypassing the augmentation) and explicit construction of the “linearized complex” $(A^\varepsilon, d^\varepsilon)$ for the algebra $A = A(L)$ corresponding to a front diagram L equipped with a normal ruling.

2.7 Normal rulings and estimates for the Thurston-Bennequin number

Within any fixed topological knot type K there exist Legendrian knots $\ell \in K$ with $TB(\ell)$ negative of arbitrarily large magnitude. However, the set $\{TB(\ell) | \ell \in K\}$ is bounded from above. For instance, there are estimates in terms of the two-variable knot polynomials ([FT], [CG], [Ta], [Ng1]):

$$TB(\ell) \leq -\deg_a F_K - 1 \quad (1)$$

$$TB(\ell) + |R(\ell)| \leq -\deg_a P_K - 1$$

where $F_K, P_K \in \mathbb{Z}[a^{\pm 1}, z^{\pm 1}]$ denote the Kauffman and HOMFLY polynomials respectively (see [R] for the particular conventions). It was conjectured in [F] that a 1-graded normal ruling exists if and only if the estimate (1) is sharp. This follows from a stronger relationship.

Proposition 2.8 ([R]) *The coefficient of a^0 in $a^{TB(\ell)+1}F_K$ (resp. $a^{TB(\ell)+1}P_K$) is given by $\sum_r z^{j(r)}$ where the sum is over all 1-graded (resp. 2-graded) normal rulings r of ℓ and*

$$j(r) = \#\{\text{switching crossings}\} - \#\{\text{right cusps}\} + 1.$$

It was shown in [ChP] that in general the ρ -graded *ruling polynomial*, $R_\ell^\rho(z) = \sum_r z^{j(r)}$ where the sum is over all ρ -graded rulings, is a Legendrian isotopy invariant. Proposition 2.8 shows that in the cases $\rho = 1, 2$, R^ρ depends only on $TB(\ell)$ and the underlying topological knot type. In contrast, R^0 can distinguish between knots with identical classical invariants [ChP].

The following is a simple corollary of Theorem 2.4 and Proposition 2.8.

Corollary 2.1 *If ℓ possesses a generating family then $TB(\ell)$ is maximal within the underlying topological knot type of ℓ .*

3 Construction of a generating family of functions for a front diagram with a ruling.

Let L be a front diagram in the xz plane equipped with a \mathbb{Z} -graded normal ruling R . We assume L generic, that is, all cusps and crossings have different x coordinates. Cusps cut L into pieces which we call strands. The condition of being \mathbb{Z} -graded includes an assumption that for every strand S of L , an integer $\text{ind}(S)$ (called the index of S) is assigned such that of two strands meeting at a cusp, the upper one has the index one more than the lower one. The ruling R may be regarded as a subset of the set of crossings (the set of switches), and for each $r \in R$ the strands crossing at r have the same index. We may assume that all the indices are positive (otherwise add the same constant to all of them). Choose a number N exceeding all the indices.

We will construct a Morse family of functions, $f_t: \mathbb{R}^N \rightarrow \mathbb{R}$, such that L and R are the diagram of critical values and the Morse complex ruling corresponding to $\{f_t\}$ in the sense explained in 2.5. The conditions imposed above on the indices guarantee that functions f_t will not have either local maxima or local minima; in particular, all the level surfaces $f_t = C$ are connected.

Our family will have an additional property that for t not equal to the x coordinates of cusps and not belonging to small neighborhoods of the x coordinates of crossings from R , the Morse complex of F_t has the simplest possible structure. Namely, the critical points of F_t are arranged into pairs, according to the ruling, and in every such pair, the indices of critical points differ by one; we state that for every pair there will be precisely one gradient trajectory joining these two points, and there will be no other gradient trajectories joining the critical points of adjacent indices.

For $t < -C$ (where C is a positive number such that L is contained in the domain $|x| < C$), we put $f_t(x_1, \dots, x_N) = x_N$. Moving t to the right, we reach the leftmost cusp of L . At this moment, we create a pair of critical points of appropriate indices with appropriate critical values such that the point with the smaller value (= the smaller index) lies precisely above the other point (here and below, we regard the direction of the x_N axis as vertical and upward). Moving further, we create, in a similar way, pairs of critical points at every left cusp in such a way that every new pair is located at a big distance (in terms of x_1, \dots, x_{N-1} from the previous pairs. When t grows, but does not reach an x coordinate of a crossing from R , the critical values change according to the z coordinates of points of L , and the pairs of critical points remain on lines parallel to the x_N axis, and the gradient trajectories joining critical points of adjacent indices are all straight vertical lines. Nothing changes at crossings not belonging to R (no crossings involves points from the same pair). If moving t to the right we arrive at a right cusp, then the strands meeting at the cusp correspond to each other with respect to the ruling R ; hence the corresponding critical points form a removable pair, and we remove them. It remains to explain what happens when we arrive at a crossing r belonging to R . According to the definition given in 2.4, three cases are possible (see Figure 4.)

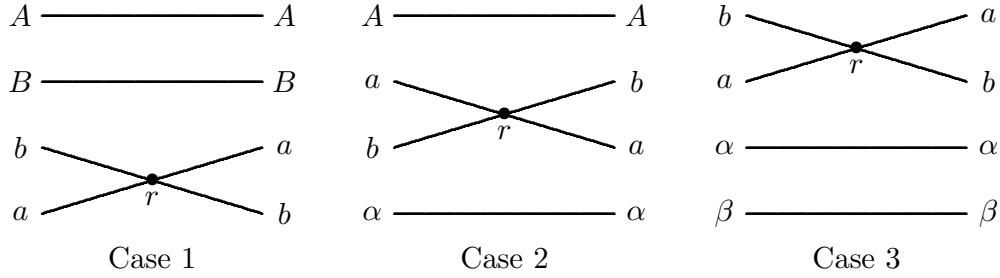


Figure 4: Three cases for a switch of a ruling

In Case 1, fragments of pairs of paths forming the ruling (directed from the left to the right) are arb , AA and bra , BB ; in Case 2 these fragments are arb , AA and bra , $\alpha\alpha$ and in Case 3 arb , $\alpha\alpha$ and bra , $\beta\beta$. In all cases, a and b have the same index, say, k ; then A and B have index $k + 1$, and α and β have index $k - 1$.

The family of function corresponding to Case 1 is presented in Figure 5. The upper diagram show the function $F_{t'}$ for some $t' < t$. The critical points

A and a are joined by a gradient trajectory, and so are the critical points B and b . The differential of the Morse complex acts as $A \mapsto a$, $B \mapsto b$. Then the two pairs of critical points are moved to each other. In the next diagram, the differential of the Morse complex becomes $A \mapsto a$, $B \mapsto b + a$ which is the same as before modulo a triangular transformation: the critical value at b exceeds the critical value at a . Then these critical values are swapped (it is the crossing (the middle diagram corresponds to the function F_t where t is the x coordinate of r), and the differential of the Morse complex (the same as before), $A + B \mapsto b$, $B \mapsto a + b$, becomes, after a triangular transformation, $A \mapsto b$, $B \mapsto a$ (now, the critical value at a exceed that at b).

At last, the two new pairs of critical points, A, b and B, a are moved apart. (By the way, the transition from the second from the bottom diagram in Figure 5, left to the bottom one may seem not clear in Figure 5, left; the reason is that our drawings are 2-dimensional, while the actual number of variables is at least 3; the level surface of the point a shown as a thick curve at the second from the bottom diagram is, actually, connected, and we can move the critical points a and B along this surface in an arbitrary way.) The bottom diagram corresponds to the function $f_{t''}$ with a $t'' > t$; here, again, all gradient trajectories between critical points of adjacent indices are vertical lines. Remark, that our functions may have many other (pairs of) critical points, but all of them stay frozen in our deformation.

In Case 2, the family of functions is shown in Figure 5, right. The critical points A, a, b, α have indices $k + 1, k, k, k - 1$. In the first diagram (corresponding to a $t' < t$), the Morse differential acts as $A \mapsto a$, $b \mapsto \alpha$. Then we move the two pairs of critical points to each other, and the differential becomes $A \mapsto a + b \mapsto 0$, $b \mapsto \alpha$ which is triangular equivalent to the previous differential while the value at a exceeds the value at b . When this values are swapped (this happens when we pass from the second diagram to the forth one), then this differential becomes triangular equivalent to $A \mapsto b$, $a \mapsto \alpha$, and it remain to move the pairs A, b and a, α of critical points apart (see the bottom diagram which corresponds to the function $F_{t''}$, $t'' > t$).

Case 3 is symmetric to Case 1 and is also illustrated by Figure 5, left (it is sufficient to reflect all the diagrams in horizontal lines and replace A and B by α and β).

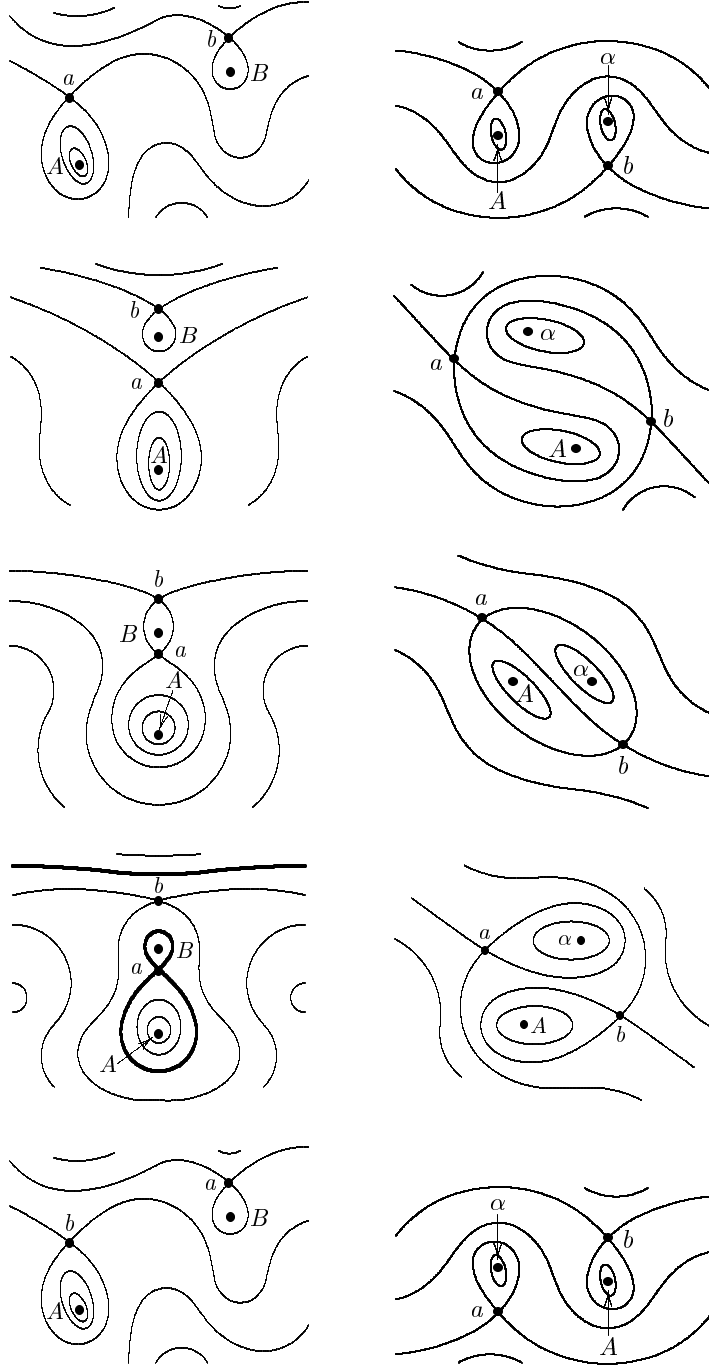


Figure 5: Family of functions (Case 1 on the left, case 2 on the right)

4 Relations between rulings, augmentations, and the linearized complex

4.1 Splash construction

Proofs of many results of the Legendrian knot theory, including Theorem 2.7, depend on (various versions of) the “splash construction” which first appeared in [F]. The goal of this construction is to modify an xy -diagram of a Legendrian knot in such a way that the differential of the Chekanov-Eliashberg DGA is described by explicit formulas; this is achieved at the expense of increasing the number of crossings. We describe here a version of the splash construction.

We begin with a pair of diagrams of a Legendrian knot such that Ng’s resolution procedure (Section 2.3) has been applied so that the xz and xy diagram have similar appearance. The xy -diagram is cut by vertical lines into “laminated zones” separated by “crossings” and “cusps”. Each laminated zone consists of several horizontal strands stretched between the vertical boundaries of the zone. The laminated zones are separated by inserts of four types (“crossings,” “left cusps,” “right cusps,” or “parallel lines”) shown in Figure 6, left.

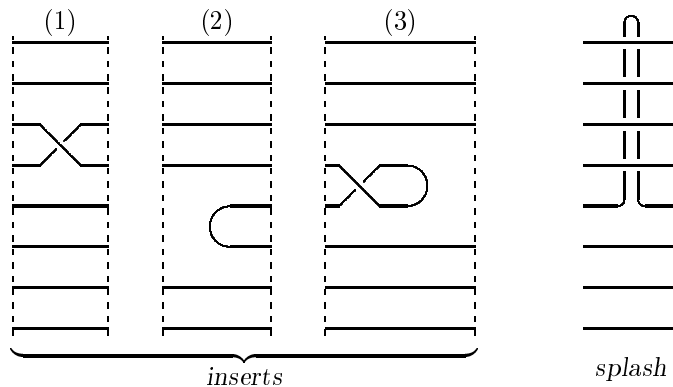


Figure 6: Inserts and splashes.

In the “laminated zones” we make “splashes” shown in Figure 6, right. We add a splash to each strand in the zone starting from the second top strand and ending with the bottom strand. Figure 7, right, shows the diagram of Figure 1, right, modified by splashes. As to the xz -diagram, the splashes

leave it almost unchanged: low steep steps appear on its strands (Figure 7, left).

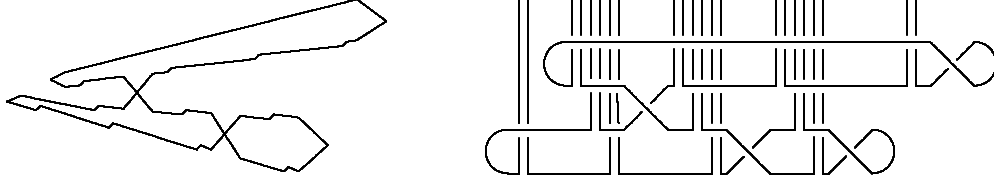


Figure 7: Splash construction.

To describe the Chekanov-Eliashberg DGA of the splashed diagram, we need notations for the crossings in the latter. Numerate the laminated zones from the left to the right by the numbers $1, 2, \dots, N$, and the inserts, also from the left to the right, by the numbers $1, 2, \dots, N + 1$; thus, the insert to the left of the laminated zone number m has also number m . The strands within a laminated zone we label, from the top to the bottom, by elements of an ordered set, usually (but not always, see below), by the numbers $1, 2, \dots, n$. The crossings of the splash on the j -th strand with the i -th strand ($i < j$) we denote as $x_{m;ij}^\pm$, with $x_{m;ij}^-$ to the left of $x_{m;ij}^+$. The crossing within m -th insert we denote as y_m , if this crossing arises from a crossing of the front diagram, and as z_m , if this crossing arises from a right cusp.

Adding splashes in the laminated zones has a simplifying effect on d . The polygons that we need to count become trapped between two adjacent laminated zones so that d of a generator coming from the m -th laminated zone is contained in the sub-algebra generated by generators from the m -th and $(m - 1)$ -st laminated zones and m -th insert.

Warning. Although it may cause confusion, when providing formulas for d it is convenient to allow a different labeling of the strands (and hence a different labeling of the generators in laminated zones). The differential $dx_{m;ij}^-$ depends only on the m -th insert and consists of polynomials in the $x_{m;ij}^\pm$, $x_{m-1;ij}^+$ and z_m or y_m . When discussing $dx_{m;ij}^-$, we will use the following numeration of strands in the m -th and $(m - 1)$ -st zones. If the m -th insert contains a crossing y_m or simply parallel lines, then we numerate the strands in the both zones (ordered from the top to the bottom) by the same numbers $1, 2, \dots, n$. If the m -th insert contains a left cusp we numerate the strands in the m -th zone by the numbers $1, 2, \dots, n$, and in the $(m - 1)$ -st zone by

the same numbers except the numbers k and $k + 1$ of the strands involved in the corresponding left cusp of the front diagram. Similarly, if the m -th insert contains a crossing z_m coming from a right cusp, then we numerate the strands in the $(m - 1)$ -st zone by the numbers $1, 2, \dots, n$ and the strands in the m -th zone by the same numbers except the numbers k and $k + 1$ of the strands involved in the right cusp. However, when discussing $dx_{m-1;ij}^-$ we use a (possibly different) labeling of the strands depending on what the $m - 1$ -th insert is. In the formula for $dx_{m;ij}^+$ we always use the standard labeling of $1, \dots, n$ increasing from top to bottom.

Proposition 4.1 (i) *If the m -th insert contains a crossing y_m between k -th and $(k + 1)$ -st strands, then*

$$\begin{aligned} dx_{m;ij}^+ &= \sum_{i < s < j} x_{m;is}^+ x_{m;s,j}^+, \quad dy_m = x_{m-1;k,k+1}^+, \\ dx_{m;ij}^- &= x_{m;ij}^+ + \sum_{i < s < j} [x_{m;is}^+ x_{m;s,j}^- + x_{m;is}^- \tilde{x}_{m-1;s,j}^+] + \tilde{x}_{m-1;i,j}^+, \end{aligned} \quad (1)$$

where $\tilde{x}_{m-1;k,k+1}^+ = 0$, and, for $u < v$, $(u, v) \neq (k, k + 1)$,

$$\tilde{x}_{m-1;uv}^+ = x_{m-1;\bar{u}\bar{v}}^+ + \begin{cases} y_m x_{m-1;uv}^+, & \text{if } u = k + 1, \\ x_{m-1;uv}^+ y_m, & \text{if } v = k, \end{cases}$$

where, in turn, $\bar{w} = w$, if $w \neq k, k + 1$, $\bar{k} = k + 1$, $\overline{k + 1} = k$.

(ii) *If the m -th insert contains a left cusp, then the differentials $dx_{m;ij}^\pm$ are expressed by the same formulas (1) with a simpler formula for $\tilde{x}_{m-1;uv}^+$:*

$$\tilde{x}_{m-1;uv}^+ = \begin{cases} x_{m-1;uv}^+, & \text{if } u \neq k, k + 1, \ v \neq k, k + 1, \\ 1, & \text{if } u = k, \ v = k + 1 \\ 0 & \text{otherwise.} \end{cases}$$

(iii) *If the m -th insert contains right cusp with crossing z_m , then, for $i \neq k, k + 1$, $j \neq k, k + 1$,*

$$\begin{aligned} dx_{m;ij}^+ &= \sum_{\substack{i < s < j \\ s \neq k, k+1}} x_{m;is}^+ x_{m;s,j}^+, \quad dz_m = 1 + x_{m-1;k,k+1}^+, \\ dx_{m;ij}^- &= x_{m;ij}^+ + \sum_{\substack{i < s < j \\ s \neq k, k+1}} [x_{m;is}^+ x_{m;s,j}^- + x_{m;is}^- \tilde{x}_{m-1;s,j}^+] + \tilde{x}_{m-1;i,j}^+, \end{aligned}$$

where

$$\tilde{x}_{m-1;uv}^+ = x_{m-1;uv}^+$$

unless $u < k, k + 1 < v$ in which case

$$\begin{aligned} \tilde{x}_{m-1;uv}^+ &= x_{m-1;uv}^+ + x_{m-1;u,k+1}^+ x_{m-1;k,v}^+ \\ &\quad + x_{m-1;uk}^+ z_m x_{m-1;k,v}^+ + x_{m-1;u,k+1}^+ z_m x_{m-1;k+1,v}^+ + x_{m-1;uk}^+ z_m^2 x_{m-1;k+1,v}^+ \end{aligned}$$

(iv) If the m -th insert is simply n parallel lines (does not correspond to any crossing or cusp on the front diagram), then

$$\begin{aligned} dx_{m;ij}^+ &= \sum_{i < s < j} x_{m;is}^+ x_{m;s,j}^+ \\ dx_{m;ij}^+ &= x_{m;ij}^+ + \sum_{i < s < j} x_{m;is}^+ x_{m;s,j}^- + x_{m-1;ij}^+ + \sum_{i < s < j} x_{m;is}^- x_{m-1;s,j}^+ \end{aligned}$$

Remark that in all cases the first formula has a short matrix form: if X_m^+ is the (strictly upper triangular) matrix with the entries $x_{m;ij}^+$, then $dX_m^+ = (X_m^+)^2$. The other formulas in Proposition 4.1 also have matrix presentations;

$$dX_m^- = X_m^+(I + X_m^-) + (I + X_m^-)\tilde{X}_{m-1}^+$$

where the matrix \tilde{X}_{m-1}^+ depends on the location and type of the singularity within the m -th insert. For instance, if the insert is a crossing between the k and $k+1$ -st strands then $\tilde{X}_{m-1}^+ = A\hat{X}_{m-1}^+A^{-1}$ where \hat{X}_{m-1}^+ is the matrix with the entries $x_{m-1;ij}^+$ except for the entry $x_{m-1;k,k+1}^+$ which is removed, and A being the block diagonal matrix with the block in the k -th and $(k+1)$ -st rows and columns being $\begin{bmatrix} 0 & 1 \\ 1 & y_m \end{bmatrix}$ and all other diagonal blocks being just $[1]$.

4.2 The standard augmentation for a splashed diagram with a ruling.

Suppose that a front diagram of Figure 1 has a normal ruling. We will give a canonical construction of an augmentation for a Chekanov–Eliashberg DGA of the corresponding splashed diagram; if the ruling is graded (ρ -graded) then the augmentation will also be graded (ρ -graded). (This construction is a modification of the construction from [F] where the splashed diagram was defined in a slightly different way.) Denote by τ the involution of the set of strands between m -th and $(m+1)$ -st inserts corresponding to our ruling.

Then we put $\varepsilon(x_{m;ij}^+) = 1$, if $i < \tau(i) = j$. If the crossing y_m of the strands numbers $k, k+1$ (in the numeration to the right of the crossing) is a switch of the ruling, then we also put $\varepsilon(y_m) = 1, \varepsilon(x_{m;k,k+1}^-) = 1$ and, in addition to that, $\varepsilon(x_{m;\tau(k+1),\tau(k)}^-) = 1$, if $\tau_m(k+1) < \tau(k)$ (that is, in the first and the third of the three cases allowed by the normality condition, as shown in Figure 4). For all other crossings of the splashed diagram (in particular, for all the crossings z_m), we put the value of ε equal to 0.

Proposition 4.2 *The function ε described above, provides an augmentation of the Chekanov–Eliashberg DGA corresponding to the splashed diagram. Moreover, this augmentation is graded (ρ -graded), if so is the normal ruling used in the construction.*

Proof is essentially contained in [F]; it can also be easily derived from the formulas for the differential d given above.

For a given augmentation ε , it is easy to describe explicitly the corresponding “linearized differential” d^ε . Namely, to find $d^\varepsilon b^\varepsilon$ for a crossing b , one should replace in the expression for db each crossing a by the binomial $\varepsilon(a) + a^\varepsilon$, perform all the multiplications, and cross out all the monomials of degree > 1 (constant terms will cancel, because ε is an augmentation). Applying this procedure to our DGA as described above, we get the following formulas for d^ε (we abbreviate the notation a^ε , where a is a crossing, to a).

Proposition 4.3 (i)

$$d^\varepsilon x_{m;ij}^+ = \left[x_{m;i\tau(j)}^+, \text{ if } i < \tau(j) < j \right] + \left[x_{m;\tau(i)j}^+, \text{ if } i < \tau(i) < j \right].$$

(ii) *If the m -th insert contains a crossing y_m between k -th and $(k+1)$ -st strand, and this crossing is not a switch, then $d^\varepsilon y_m = x_{m-1;k,k+1}^+$ and*

$$\begin{aligned} d^\varepsilon x_{m;ij}^- = x_{m;ij}^+ &+ \left[x_{m-1;\overline{i}\overline{j}}^+, \text{ if } (i,j) \neq (k,k+1) \right] \\ &+ \left[x_{m;i\tau(j)}^-, \text{ if } i < \tau(j) < j \right] + \left[x_{m;\tau(i)j}^-, \text{ if } i < \tau(i) < j \right] \end{aligned}$$

where (as above) $\overline{w} = w$, if $w \neq k, k+1$, $\overline{k} = k+1, \overline{k+1} = k$.

(iii) If the m -th insert contains a crossing y_m between k -th and $(k+1)$ -st strand, and this crossing is a switch, then again $d^\varepsilon y_m = x_{m-1;k,k+1}^+$ and

$$\begin{aligned}
d^\varepsilon x_{m;ij}^- &= x_{m;ij}^+ + [x_{m-1;ij}^+, \text{ if } (i,j) \neq (k,k+1)] \\
&\quad + [x_{m-1;kj}^+, \text{ if } i = k+1] + [x_{m-1;i,k+1}^+, \text{ if } j = k] \\
&\quad + [x_{m-1;\tau(k)j}^+, \text{ if } i = \tau(k+1) < \tau(k) < j] \\
&\quad \quad + [x_{m-1;\tau(k),k+1}^+, \text{ if } i = \tau(k+1) < \tau(k) < k = j] \\
&\quad + [x_{m;\tau(i)j}^-, \text{ if } i < \tau(i) < j] + [x_{m;i\tau(j)}^-, \text{ if } i < \tau(j) < j] \\
&\quad + [x_{m;i,k+1}^-, \text{ if } i < k = \tau(j) < j] + [x_{m;i,k}^-, \text{ if } i < k, k+1 = \tau(j) < j] \\
&\quad + [x_{m;i\tau(k+1)}^-, \text{ if } i < \tau(k+1) < j = k] \\
&\quad + [y_m, \text{ if } i = k+1, j = \tau(k+1) \text{ or } i = \tau(k), j = k]
\end{aligned}$$

(iv) If the m -th insert contains no crossings, then $d^\varepsilon x_{m;k,k+1}^- = x_{m;k,k+1}^+$ and, for $(i,j) \neq (k,k+1)$,

$$\begin{aligned}
d^\varepsilon x_{m;ij}^- &= x_{m;ij}^+ + [x_{m-1;ij}^+, \text{ if } i \neq k, k+1, j \neq k, k+1] \\
&\quad + [x_{m;\tau(i)j}^-, \text{ if } i < \tau(i) < j] + [x_{m;i\tau(j)}^-, \text{ if } i < \tau(j) < j]
\end{aligned}$$

(v) If the m -th insert contains a crossing z_m , then, for $i \neq k, k+1, j \neq k, k+1$,

$$\begin{aligned}
d^\varepsilon x_{m;ij}^- &= x_{m;ij}^+ + x_{m-1;ij}^+ \\
&\quad + [x_{m;\tau(i)j}^-, \text{ if } i < \tau(i) < j] + [x_{m;i\tau(j)}^-, \text{ if } i < \tau(j) < j]
\end{aligned}$$

4.3 Algorithmic computation of the linearized homology for a front diagram with a ruling

The complex with the differential d^ε described in Proposition 4.3 turns out to be homotopy equivalent to a much smaller complex which can be described by an explicit and relatively simple algorithm in terms of the initial front diagram and a normal ruling. The additive generators of this complex correspond to the crossings and right cusps of the front diagram.

Let \mathcal{C}^ε be the complex with the basis $\{x_{m;ij}^+, x_{m;ij}^-, y_m, z_m\}$ and the differential d^ε as described in Proposition 4.3. Furthermore, let $\mathcal{C}_-^\varepsilon$ be the subspace

of \mathcal{C}^ε with the basis $\{x_{m;ij}^-\}$ and $\mathcal{C}_+^\varepsilon$ be the quotient of \mathcal{C}^ε over the subspace with the basis $\{x_{m;ij}^-, y_m, z_m\}$; thus $\{x_{m;ij}^+\}$ is a basis in $\mathcal{C}_+^\varepsilon$.

Lemma 4.1 (i) *The differential d^ε maps $\mathcal{C}_-^\varepsilon$ isomorphically onto $d^\varepsilon(\mathcal{C}_-^\varepsilon)$.*
(ii) *The restriction of the projection $\mathcal{C}^\varepsilon \rightarrow \mathcal{C}_+^\varepsilon$ to $d^\varepsilon(\mathcal{C}_-^\varepsilon)$ is an isomorphism.*
(iii) *Cosets of y_m, z_m form a basis in $\mathcal{C}_0^\varepsilon = \mathcal{C}^\varepsilon / [\mathcal{C}_-^\varepsilon \oplus d^\varepsilon(\mathcal{C}_-^\varepsilon)]$.*

Proof The composition

$$\mathcal{C}_-^\varepsilon \xrightarrow{d^\varepsilon} d^\varepsilon(\mathcal{C}_-^\varepsilon) \xrightarrow{\text{projection}} \mathcal{C}_+^\varepsilon$$

maps $x_{m;ij}^-$ onto $x_{m;ij}^+ + \dots$ where the dots stand for terms of the form $x_{m-1;uv}^+$ or $x_{m;uv}^+$ where in the second case $u - v > i - j$; this means that the composition is upper triangular with respect to a certain partial ordering of bases. The statements of the Lemma follow.

Lemma 4.1 implies that $\mathcal{C}_-^\varepsilon \oplus d^\varepsilon(\mathcal{C}_-^\varepsilon)$ is an acyclic subcomplex of \mathcal{C}^ε , so $\mathcal{C}_0^\varepsilon = \mathcal{C}^\varepsilon / [\mathcal{C}_-^\varepsilon \oplus d^\varepsilon(\mathcal{C}_-^\varepsilon)]$ is a complex homotopy equivalent to \mathcal{C}^ε . Moreover, this complex is additively generated by y_m and z_m , that is, by crossings and right cusps of the given front diagram. Proposition 4.3 gives rise to an explicit description of the differential d_0^ε of the complex $\mathcal{C}_0^\varepsilon$; moreover, it turns out that d_0^ε depends only on the front diagram and the normal ruling in this front diagram. Below we give a description of this differential in the form of an algorithm applied to a generic front diagram with a normal ruling. Remark that if the rotation number of the given front diagram is 0 and the normal ruling is graded, then the differential has degree -1 ; a similar statement is true in the ρ -graded case.

To compute d_0^ε of a crossing or a right cusp of the given front diagram we start with a vertical segment connecting the two involved strands directly to the left of the singularity. Then we push the segment to the left in such a way that its ends are always contained in the diagram, and every time a crossing occurs on the diagram the segment will either contribute that crossing to the differential or it will not, according to the following rules.

At a non-switch crossing between the k -th and $(k+1)$ -st strands, let $\tau(k)$ and $\tau(k+1)$ be the strands corresponding to k and $k+1$ in the ruling (to the right of the crossing). Then segments with the top at $\tau(k+1)$ and the bottom at k and segments with the top at $k+1$ and the bottom at $\tau(k)$ contribute the crossing to the differential; others do not.

At a switched strand things are even simpler. A segment will contribute the switched crossing to the differential, if and only if on one end of this

segment we have one of the crossing strands (k or $k + 1$), and on the other end we have one of the two companion strands ($\tau(k)$ or $\tau(k + 1)$).

After a crossing segment(s) will prolong to the left side of the crossing, and possibly multiply into more segments, according to the following rules.

If the segment approaches (the vertical level of) a left cusp, and at least one of its ends belongs to a strand involved in the cusp, this segment disappears; otherwise, it passes through the cusp without any changes. If the segment approaches a non-switch crossing, and both ends belong to the strands involved in the crossing, then this segment disappears; otherwise it passes through the crossing without changes (the ends follow the strands). In the case when the segment approaches a switch crossing, the rules are more complicated; in this case, it will be convenient to perform the modification of the segment in two steps: first, we replace the segment approaching the switch crossing by 0, 1, or 2 segments immediately to the left of the crossing (Step 1).

There are 3 types of switches (as in figure 4). For the case 2 switch Step 1 completes the prolonging process. For case 1 or case 3 switches, some of the segments appeared after Step 1, (without being eliminated themselves) give birth to additional segments (Step 2). Remark that in our notations for segments we always list the ends starting with the top one; thus, speaking of a segment $[\ell, m]$ we always mean that $\ell < m$. The strands involved in the crossing always denoted as k and $k + 1$. Thus, $\tau(k + 1) < \tau(k) < k$, or $\tau(k) < k < k + 1 < \tau(k + 1)$, or $k + 1 < \tau(k + 1) < \tau(k)$ (the normality condition).

STEP 1

Segment	such that	becomes
$[k, k + 1]$	$\{\ell, m\} \cap \{k, k + 1\} = \emptyset$ $\ell \neq k$ $m \neq k + 1$	\emptyset
$[\ell, m]$		$[\ell, m]$
$[\ell, k]$		$[\ell, k], [\ell, k + 1]$
$[\ell, k + 1]$		$[\ell, k + 1]$
$[k, m]$		$[k, m]$
$[k + 1, m]$		$[k, m], [k + 1, m]$

STEP 2 (Case 1 and Case 3 switch only)

Segment	such that	gives birth to
$[\ell, \tau(k)]$	$\ell < \tau(k+1) < \tau(k)$	$[\ell, \tau(k+1)]$
$[\tau(k+1), m]$	$\tau(k+1) < \tau(k) < m$	$[\tau(k), m]$

Thus, after passing a crossing, a segment becomes a (maybe, empty) set of at most 4 segments. The maximal amount of segments appears in two cases: if $k+1 < \tau(k+1) < \tau(k)$, then the segment $[k+1, \tau(k)]$ becomes the quadruple $[k, \tau(k)]$, $[k+1, \tau(k)]$, $[k, \tau(k+1)]$, $[k+1, \tau(k+1)]$; if $\tau(k+1) < \tau(k) < k$, then the segment $[k, \tau(k+1)]$ becomes the same quadruple. These transformations are shown in Figure 8 (where crossings represent switches).

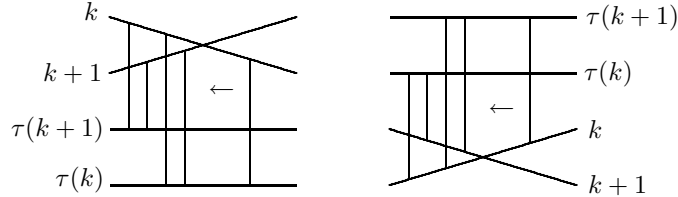


Figure 8: One segment may produce as many as four segments

Thus, we have completed a construction, for a front diagram of a Legendrian knot with a graded normal ruling a complex additively generated by crossings and right cusps of the diagram; its homology coincides with the linearized homology of an augmented Chekanov-Eliashberg DGA of an xy -diagram of a Legendrian isotopic Legendrian knot. For a front diagram L with a graded normal ruling τ , denote this complex as $\mathcal{C}(L, \tau)$.

Example. We consider a class of front diagrams first introduced in [Ng2] which always admit a normal ruling.

Definition 4.1 A *Mondrean diagram* is a collection of horizontal and vertical line segments in the xz -plane, so that

- (i) The endpoints of vertical segments lie on the interior of horizontal segments.
- (ii) There are no other points of intersection between different segments.

Given a Mondrean diagram θ a front diagram $L(\theta)$ is formed in two steps:

- (1) Replace each horizontal segment with a standard Legendrian unknot (a front with two cusps and no crossings). These should be thin enough so they are all disjoint.
- (2) Vertical segments become crossings between the two adjacent pieces of the Legendrian unknots corresponding to the two horizontal segments which the vertical segment connects (see Figure 9).

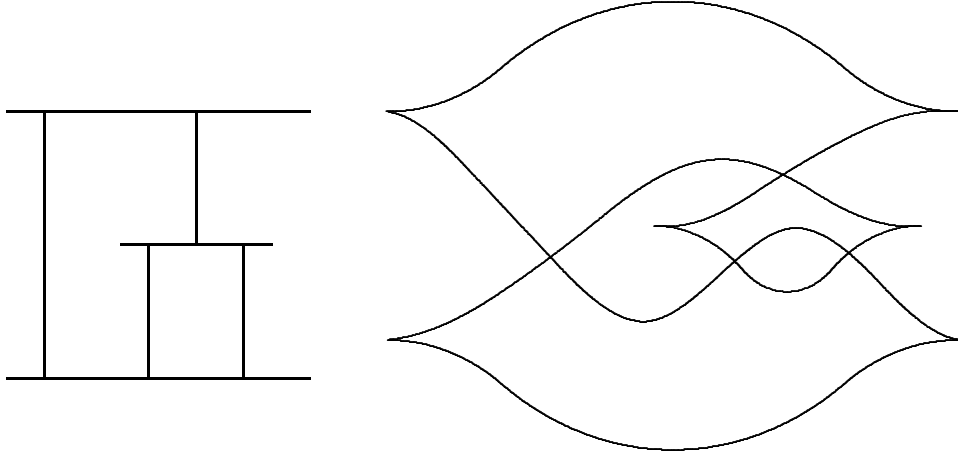


Figure 9: A Mondrean diagram and a corresponding front diagram

Remark. Ng shows that within any alternating knot type there exists a Legendrian representative whose front diagram is obtained by the above process [Ng2]. See [K1] for a related result involving the wider class of $+-$ adequate links.

Note that a front $L(\theta)$ obtained in this manner always has a normal ruling τ where every crossing is switched. Using the algorithm described above we may explicitly compute the differential in $\mathcal{C}(L(\theta), \tau)$.

To notate generators, label horizontal segments of θ as h_1, h_2, \dots, h_M in decreasing order of their z -coordinates. Given two horizontal segments h_i, h_j with $i < j$ there will be some number of vertical segments (possibly 0) connecting them. Label these vertical segments from right to left as $v_{ij}^1, v_{ij}^2, \dots, v_{ij}^N$. There are one-to-one correspondences:

$$\begin{aligned} \text{horizontal segments} &\leftrightarrow \text{right cusps} \\ \text{vertical segments} &\leftrightarrow \text{crossings} \end{aligned}$$

according to which we will use the same notations for the generators of $\mathcal{C}(L(\theta), \tau)$. Our formulas are simplified by treating the lower indices of vertical segments as an unordered pair, so that $v_{ij}^n = v_{ji}^n$.

Lemma 4.2 *The differential in $\mathcal{C}(L(\theta), \tau)$ is given by the formulas*

$$(i) \ d_0^\varepsilon h_i = \sum_j \sum_{n \text{ odd}} v_{ij}^n, \quad (ii) \ d_0^\varepsilon v_{ij}^n = 0.$$

Proof. In the first step of forming a front diagram from θ each horizontal segment h_i is expanded into two arcs joined at their endpoints by cusps. Let t_i and b_i denote the top arc and bottom arc respectively. Throughout the front diagram these arcs will be companion strands according to the ruling τ . Away from x -values of crossings these arcs constitute all of $L(\theta)$. In computing d_0^ε we manipulate vertical segments with endpoints on $L(\theta)$. We label such a segment, for instance, as $[b_i, t_j]$ if the segment has part of b_i for an upper endpoint and part of t_j for a lower endpoint.

For the first formula of the Lemma we begin with a segment $[t_i, b_i]$ appearing directly to the left of the right cusp h_i . All of the switches of τ are Case 2. Due to this fact, the only types of segments which can be created during the prolonging process will be of the form $[t_i, b_i]$, $[t_i, t_k]$, $[b_j, b_i]$, or $[b_j, t_k]$. Further notice that if a segment is created with its bottom (resp. top) endpoint lying on some t_k (resp. b_j) then that segment can only prolong to other segments whose bottom (resp. top) endpoint lies on t_k (resp. b_j).

When we arrive at some crossing v_{jk}^n each segment of the form $[t_j, b_j]$, $[t_k, b_k]$, $[t_j, t_k]$, $[b_j, b_k]$ contributes v_{jk}^n to $d_0^\varepsilon h_i$, so we need to determine the number of such segments $\mod 2$. In the case, that $i \neq j, i \neq k$ there are no such segments. Consider now the case of a crossing v_{ij}^n with $i < j$. If $j < i$ a similar argument holds.

The only possible segments which will contribute v_{ij}^n to $d_0^\varepsilon h_i$ are of the form $[t_i, b_i]$ and $[t_i, t_j]$. During the prolonging process every time we pass a crossing v_{ij}^l a new segment of the form $[t_i, t_j]$ appears. ($[t_i, b_i]$ prolongs to a copy of $[t_i, b_i]$ and $[t_i, t_j]$ to the left of the crossing, while $[t_i, t_j]$ prolongs to $[t_i, t_j]$.) The number of segments of these two forms is not affected when prolonging through other types of crossings. Since we count segments $\mod 2$, (i) follows.

For $d_0^\varepsilon v_{ij}^n$ we begin with a segment $[b_i, t_j]$ directly to the left of the crossing. Such a segment remains of the form $[b_i, t_j]$ as it is pushed to the left according

to the algorithm, and hence will not contribute any terms to the differential.

Since every crossing is switched the ruling τ is rarely graded, and in the case that it is computation of the corresponding linearized homology groups follows immediately from Sabloff's Duality result, [S1]. In general, if we take ρ so that τ is ρ -graded then in the resulting \mathbb{Z}_ρ grading on $\mathcal{C}(L(\theta), \tau)$ all crossings have degree 0 and all cusps have degree 1. Homology computations follow easily from the Lemma.

Proposition 4.4 *Suppose the Mondrean diagram θ is connected. Then with τ as above,*

$$\dim_{\mathbb{Z}_2} HC(L(\theta), \tau) = \# \text{crossings} - \# \text{right cusps} + 2.$$

If τ is ρ -graded with $\rho \neq 1$ then

$$\begin{aligned} \dim_{\mathbb{Z}_2} H_0 \mathcal{C}(L(\theta), \tau) &= \# \text{crossings} - \# \text{right cusps} + 1, \\ \dim_{\mathbb{Z}_2} H_1 \mathcal{C}(L(\theta), \tau) &= 1 \\ \dim_{\mathbb{Z}_2} H_l \mathcal{C}(L(\theta), \tau) &= 0, \quad l \neq 0, 1. \end{aligned}$$

Remark. The proposition shows the linearized groups arising from this maximally switched ruling are not so useful in distinguishing between Mondrean diagram fronts with the same underlying knot type. This is because the quantity

$$\# \text{crossings} - \# \text{right cusps} + 1$$

is simply the degree of the ungraded ruling polynomial, $R_{L(\theta)}^1(z)$, and according to Proposition 2.8 R^1 depends only on the Thurston-Bennequin number and the Kauffman polynomial of the underlying knot type. Since all of the crossings are switches we cannot assign a ρ -grading to the complex $\mathcal{C}(L(\theta), \tau)$ so that crossings have degree other than zero, and in practice it is usually the grading which allows linearized homology groups to distinguish Legendrian knots with the same classical invariants.

4.4 From an augmentation to a normal ruling: a new approach

Let L be a Ng front diagram of a Legendrian knot, and let Γ be the xy -diagram obtained from L by the splash construction (see Figures 1 and 8).

We present here an algorithm which assigns to a (graded) augmentation of the Chekanov-Eliashberg DGA corresponding to Γ a (graded) normal ruling of L . This provides a new approach to the result of Fuchs, Ishkhanov and Sabloff (Theorem 2.7) and reveals its connection with Proposition 2.5 (and hence with generating families).

The Chekanov-Eliashberg DGA of Γ is described in Proposition 4.1; below we use the notations of this Proposition. Let ε be an augmentation of this DGA. Put $e_{m;ij}^\pm = \varepsilon(x_{m;ij}^\pm)$, and let E_m^\pm be the matrix with the entries $e_{m;ij}^\pm$ (that is, $E^\pm = \varepsilon(X_m^\pm)$). The matrix E_m^\pm is triangular (that is, $e_{m;ij}^\pm = 0$ for $j \leq i$). Since $\varepsilon \circ d = 0$, the relation $dX_m^+ = (X_m^+)^2$ (see Section 4.1) shows that $(E_m^+)^2 = 0$.

Let \mathcal{C}_m be the vector space with the basis $\{\sigma_i\}$ labelled with the numbers of strands in the m -th laminated zone. Put $\partial(\sigma_i) = \sum e_{m;ij}^+ \sigma_j$. Since $(E_m^+)^2 = 0$, $(\mathcal{C}_m, \partial)$ is a complex (with a triangular differential, as in Proposition 2.5). Moreover, if the rotation number of the Legendrian knot is 0 and the augmentation ε is graded, then \mathcal{C}_m is graded and $\deg \partial = -1$.

Lemma 4.3 *The complex \mathcal{C}_m is acyclic.*

Proof It is sufficient to prove that the homologies of the complexes \mathcal{C}_m and \mathcal{C}_{m-1} are the same. Since $\mathcal{C}_m = 0$ for $m < 0$, this implies the acyclicity.

If the m -th insert contains a crossing y_m , then the formula

$$dX_m^- = X_m^+(I + X_m^-) + (I + X_m^-)A\widehat{X}_{m-1}^+A^{-1}$$

(see Section 4.1) implies

$$E_m^+ = (I + E_m^-)\varepsilon(A)\varepsilon(\widehat{X}_{m-1}^+)\varepsilon(A)^{-1}(I + E_m^-)^{-1}. \quad (2)$$

But since $dy_m = x_{m-1;k,k+1}^+$ (see Part (i) of Proposition 4.1), $\varepsilon(x_{m-1;k,k+1}^+) = 0$, and hence $\varepsilon(\widehat{X}_{m-1}^+) = E_{m-1}^+$ (the matrices X_{m-1}^+ , \widehat{X}_{m-1}^+ are the same with the exception of the entry $x_{m;k,k+1}^+$), and the formula shows that the matrices E_m^+ , E_{m-1}^+ are conjugated. Hence the complexes \mathcal{C}_m and \mathcal{C}_{m-1} have the same homology.

The cases when the m -th insert contains the crossing z_m , or does not contain any crossing at all, are easier.

Let the m -th insert contain no crossings. The complex \mathcal{C}_{m-1} has the basis $\{\sigma_i \mid i \neq k, k+1\}$. Let $(\widetilde{\mathcal{C}}_{m-1}, \widetilde{\partial})$ be $(\mathcal{C}_{m-1}, \partial)$ with basis elements σ_k, σ_{k+1}

added and $\tilde{\partial}$ the same as ∂ with, additionally, $\widehat{\partial}(\sigma_k) = \sigma_{k+1}$. Then the formula (2) becomes

$$E_m^+ = (I + E_m^-) \tilde{E}_{m-1}^+ (I + E_m^-)^{-1}$$

where \tilde{E}_{m-1}^+ is the matrix of $\tilde{\partial}$. Hence, the complex \mathcal{C}_m has the same homology as the complex $\tilde{\mathcal{C}}_{m-1}$ which, in turn, has the same homology as \mathcal{C}_{m-1} .

Let the m -th insert contain the crossing z_m . Then the complex \mathcal{C}_{m-1} has two generators missing in \mathcal{C}_m : σ_k and σ_{k+1} . Since $dz_m = 1 + x_{m-1;k,k+1}^+$ (see Part (iii) of Proposition 4.1), in \mathcal{C}_{m-1} , $\partial\sigma_k = \sigma_{k+1} +$ a linear combination of σ_j with $j > k+1$. Consider the complex $\tilde{\mathcal{C}}_{m-1}$ as the quotient of the complex \mathcal{C}_{m-1} by the (acyclic) two dimensional subcomplex generated by

$$\begin{cases} \sigma_k \text{ and } \partial\sigma_k, & \text{if } \varepsilon(z_m) = 0, \\ \sigma_k + \sigma_{k+1} \text{ and } \partial\sigma_k + \partial\sigma_{k+1}, & \text{if } \varepsilon(z_m) = 1. \end{cases}$$

We can assume that the complex $\tilde{\mathcal{C}}_{m-1}$ has the same basis as \mathcal{C}_m . The last formula in Proposition 4.1 shows that

$$E_m^+ = (I + E_m^-) \tilde{E}_{m-1}^+ (I + E_m^-)^{-1}$$

and the matrix \tilde{E}_{m-1}^+ , whether $\varepsilon(z_m) = 0$ or 1 , is the matrix of the differential of $\tilde{\mathcal{C}}_{m-1}$ with respect to the basis $[\sigma_j]$, $j \neq k, k+1$. Hence, the complex \mathcal{C}_m has the same homology as the complex $\tilde{\mathcal{C}}_{m-1}$ which, in turn, has the same homology as \mathcal{C}_{m-1} .

This completes the proof of Lemma.

According to Proposition 2.5, there arises a fixed point free involution τ_m in the set of generators of the complex \mathcal{C}_m , that is, in the set of strands in the m -th laminated zone of L , such that, after a triangular basis transformation $\sigma'_i = \sigma_i + \sum_{j>i} a_{ij}\sigma_j$, the differential $\partial: \mathcal{C}_m \rightarrow \mathcal{C}_m$ acts as

$$\partial(\sigma'_i) = \begin{cases} \sigma'_{\tau_m(i)}, & \text{if } i < \tau_m(i), \\ 0, & \text{if } i > \tau_m(i). \end{cases}$$

Proposition 4.5 *The involutions τ_m form a normal ruling of L , graded, if the augmentation ε is graded.*

This completes our algorithm: it provides a (graded) ruling from a (graded) augmentation.

Proof of Proposition 4.5 Let m -th insert contain a crossing y_m . After a triangular changes of basis, the complex \mathcal{C}_{m-1} and \mathcal{C}_m have differentials, respectively,

$$\partial\sigma_i = \begin{cases} \sigma_{\tau_{m-1}(i)}, & \text{if } \tau_{m-1}(i) > i, \\ 0, & \text{if } \tau_{m-1}(i) < i, \end{cases} \quad \partial\sigma_i = \begin{cases} \sigma_{\tau_m(i)}, & \text{if } \tau_m(i) > i, \\ 0, & \text{if } \tau_m(i) < i. \end{cases}$$

The transition from \mathcal{C}_{m-1} to \mathcal{C}_m (up to an additional triangular transformation by the matrix $I + E_m^-$) is performed by the matrix $\varepsilon(A)$, that is,

$$\sigma_i \mapsto \begin{cases} \sigma_i, & \text{if } i \neq k, k+1, \\ \alpha\sigma_k + \sigma_{k+1}, & \text{if } i = k, \\ \sigma_k, & \text{if } i = k+1, \end{cases}$$

where $\alpha = \varepsilon(y_m)$. There are 6 cases.

Case 1. $\tau_{m-1}(k+1) < \tau_{m-1}(k) < k$. In \mathcal{C}_{m-1} ,

$$\partial\sigma_{\tau_{m-1}(k+1)} = \sigma_{k+1}, \quad \partial\sigma_{\tau_{m-1}(k)} = \sigma_k;$$

then in \mathcal{C}_m ,

$$\begin{aligned} \partial\sigma_{\tau_{m-1}(k+1)} &= \sigma_k, \quad \partial\sigma_{\tau_{m-1}(k)} = \alpha\sigma_k + \sigma_{k+1}, \\ \partial(\sigma_{\tau_{m-1}(k+1)} + \sigma_{\tau_{m-1}(k)}) &= (1 + \alpha)\sigma_k + \sigma_{k+1}. \end{aligned}$$

Hence, if $\alpha = 0$, then $\tau_m(k) = \tau_{m-1}(k+1)$, $\tau_m(k+1) = \tau_{m-1}(k)$ (no switch), and if $\alpha = 1$, then $\tau_m = \tau_{m-1}$ (a switch).

Case 2. $\tau_{m-1}(k) < \tau_{m-1}(k+1) < k$. In \mathcal{C}_{m-1} ,

$$\partial\sigma_{\tau_{m-1}(k)} = \sigma_k, \quad \partial\sigma_{\tau_{m-1}(k+1)} = \sigma_{k+1};$$

then in \mathcal{C}_m ,

$$\begin{aligned} \partial\sigma_{\tau_{m-1}(k)} &= \alpha\sigma_k + \sigma_{k+1}, \quad \partial\sigma_{\tau_{m-1}(k+1)} = \sigma_k, \\ \partial(\sigma_{\tau_{m-1}(k)} + \sigma_{\tau_{m-1}(k+1)}) &= (1 + \alpha)\sigma_k + \sigma_{k+1}. \end{aligned}$$

Hence, whether $\alpha = 0$ or 1 , $\tau_m(k) = \tau_{m-1}(k+1)$, $\tau_m(k+1) = \tau_{m-1}(k)$ (no switch).

Case 3. $\tau_{m-1}(k) < k$, $k+1 < \tau_{m-1}(k+1)$. In \mathcal{C}_{m-1} ,

$$\partial\sigma_{\tau_{m-1}(k)} = \sigma_k, \quad \partial\sigma_{k+1} = \sigma_{\tau_{m-1}(k+1)};$$

then in \mathcal{C}_m ,

$$\partial_{\tau_{m-1}(k)} = \alpha\sigma_k + \sigma_{k+1}, \partial(\alpha\sigma_k + \sigma_{k+1}) = 0, \partial\sigma_k = \sigma_{\tau_{m-1}(k+1)}.$$

If $\alpha = 0$, then $\tau_m(k) = \tau_{m-1}(k+1)$, $\tau_m(k+1) = \tau_{m-1}(k)$ (no switch). If $\alpha = 1$, then

$$\partial\sigma_{\tau_{m-1}(k)} = \sigma_k + \sigma_{k+1}, \partial\sigma_{k+1} = \partial\sigma_k = \sigma_{\tau_{m-1}(k+1)},$$

hence $\tau_m = \tau_{m-1}$ (a switch).

Case 4. $\tau_{m-1}(k+1) < k$, $k+1 < \tau_{m-1}(k)$. In \mathcal{C}_{m-1} ,

$$\partial\sigma_{\tau_{m-1}(k+1)} = \sigma_{k+1}, \partial\sigma_k = \sigma_{\tau_{m-1}(k)};$$

then in \mathcal{C}_m ,

$$\partial\sigma_{\tau_{m-1}(k+1)} = \sigma_k, \partial(\alpha\sigma_k + \sigma_{k+1}) = \sigma_{\tau_{m-1}(k)}.$$

Hence,

$$\partial\sigma_k = 0, \partial\sigma_{k+1} = \sigma_{\tau_{m-1}(k)}.$$

Thus, whether $\alpha = 0$ or 1 , $\tau_m(k) = \tau_{m-1}(k+1)$, $\tau_m(k+1) = \tau_{m-1}(k)$ (no switch).

Case 5. $k+1 < \tau_{m-1}(k+1) < \tau_{m-1}(k)$. In \mathcal{C}_{m-1} ,

$$\partial\sigma_k = \sigma_{\tau_{m-1}(k)}, \partial\sigma_{k+1} = \sigma_{\tau_{m-1}(k+1)};$$

then in \mathcal{C}_m ,

$$\partial(\alpha\sigma_k + \sigma_{k+1}) = \sigma_{\tau_{m-1}(k)}, \partial\sigma_k = \sigma_{\tau_{m-1}(k+1)}.$$

If $\alpha = 0$, then $\tau_m(k) = \tau_{m-1}(k+1)$, $\tau_m(k+1) = \tau_{m-1}(k)$ (no switch). If $\alpha = 1$, then $\partial(\sigma_k + \sigma_{k+1}) = \sigma_{\tau_{m-1}(k)}$, and hence $\tau_m = \tau_{m-1}$ (a switch).

Case 6. $k+1 < \tau_{m-1}(k) < \tau_{m-1}(k+1)$. In \mathcal{C}_{m-1} ,

$$\partial\sigma_k = \sigma_{\tau_{m-1}(k)}, \partial\sigma_{k+1} = \sigma_{\tau_{m-1}(k+1)};$$

then in \mathcal{C}_m ,

$$\partial(\alpha\sigma_k + \sigma_{k+1}) = \sigma_{\tau_{m-1}(k)}, \partial\sigma_k = \sigma_{\tau_{m-1}(k+1)}.$$

If $\alpha = 0$, then $\partial\sigma_{k+1} = \sigma_{\tau_{m-1}(k)}$, $\partial\sigma_k = \sigma_{\tau_{m-1}(k+1)}$; if $\alpha = 1$, then $\partial(\sigma_{k+1}) = \sigma_{\tau_{m-1}(k)} + \sigma_{\tau_{m-1}(k+1)}$. Hence, whether $\alpha = 0$ or 1 , $\tau_m(k) = \tau_{m-1}(k+1)$, $\tau_m(k+1) = \tau_{m-1}(k)$ (no switch).

In addition to this, if the m -th insert contains no crossings, then in \mathcal{C}_m , $\partial\sigma_k = \sigma_{k+1}$, and if the m -th insert contains a crossing z_m , then in \mathcal{C}_{m-1} , $\partial\sigma_k = \sigma_{k+1} +$ generators with the numbers $> k + 1$ (see the proof of Lemma 4.2). Thus, in the first case $\tau_m(k) = k + 1$, and in the second case $\tau_{m-1}(k) = k + 1$.

We see that the involution constructed satisfies the requirements of a normal ruling.

5 Generating families and the homology of the linearized Chekanov-Eliashberg DGA

Work of Traynor and collaborators [JTr] [NgTr] [Tr] has used the theory of generating families to distinguish Legendrian knots (actually usually 2-component links) with identical classical invariants. A version of their approach is the following:

Let ℓ be a Legendrian knot and $F : \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$, $f_t = F(\cdot, t)$, a linear at infinity generating family for ℓ . Define $w : \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}$ to be the *difference function*

$$w(x, y, t) = f_t(x) - f_t(y).$$

Let $\delta > 0$ small enough so that the interval $(0, \delta)$ consists entirely of regular values of w . Define the *generating family homology* of F as the grading shifted homology groups $\mathcal{G}H_*(F) = H_{*+(N+1)}(w \geq \delta, w = \delta; \mathbb{Z}_2)$. The groups $\mathcal{G}H_*(F)$ almost provide a Legendrian isotopy invariant of ℓ . Actually, as in the invariance statement of Theorem 4.4, since there is no reason that these groups need to be independent of the choice of generating family the set of all such homology groups where we are allowed to choose different generating families is what forms an invariant. Also, the linear at infinity condition needs to be generalized to “linear-quadratic at infinity” to allow stabilizations. (A *stabilization* increases the dimension of the domain of a generating family by summing with a non-degenerate quadratic form in the new variables). Stabilizations are necessary in order to lift an isotopy of Legendrian knots to a homotopy of generating families.

Proposition 5.1 (Jordan-Traynor) *The set of possible graded homology groups $\{\mathcal{G}H_*(F)\}$ where F is a linear-quadratic at infinity generating family for ℓ is an invariant of Legendrian isotopy.*

Roughly, the invariance statement is proved by examining the Morse complex of w (see 5.1 for definitions). This investigation also sheds light on a possible relationship with the Chekanov-Eliashberg DGA, so we recall it here.

Proposition 5.2 *The critical points of the function w with positive critical values correspond to the crossings of the xy diagram of ℓ . Moreover, the indices of the critical points are equal to the degrees of the crossings plus $N + 1$.*

Proof. Since

$$\frac{\partial w}{\partial x} = -\frac{\partial f_t(x)}{\partial x}, \quad \frac{\partial w}{\partial y} = \frac{\partial f_t(y)}{\partial y}, \quad \frac{\partial w}{\partial t} = \frac{\partial}{\partial t} f_t(x) - \frac{\partial}{\partial t} f_t(y),$$

the point (x, y, t) is a critical point of w if and only if (1) x and y are critical points of f_t , that is, $(f_t(x), t), (f_t(y), t) \in L$, and (2) the tangents to the strands of L at these two points are parallel. Additionally, $f_t(y) > f_t(x)$ since $w(x, y, t) > 0$. Thus, (x, y, t) corresponds to a crossing in the xy -diagram. Moreover,

$$\text{ind}_{(x,y,t)} w = \text{ind}_y f_t + (N - \text{ind}_x f_t) + e$$

where $e = 1$, if t is a local maximum of the distance between the strands of the points $(f_t(x), t), (f_t(y), t)$, and $e = 0$ if t is a local minimum of this distance. Since the index of the crossing is $\text{ind}_y f_t - \text{ind}_x f_t + e$, Proposition follows.

Sketch of invariance of $\{\mathcal{GH}_*(F)\}$ Let $\ell^s, 0 \leq s \leq 1$ be a (generic) Legendrian isotopy and suppose a generating family for ℓ^0 is chosen. After possibly stabilizing the original generating family $F^0 : \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$ we can find a homotopy $F^s, 0 \leq s \leq 1$ such that F^s is a generating family for ℓ^s . Denote by w^s the corresponding difference functions. Critical values of w^s are not allowed to cross 0 since according to the Proposition 5.1 this would correspond to a self intersection of one of the ℓ^s during the isotopy. Therefore, after choosing a suitable family of metrics, the only changes to the Morse complex will be through the birth or death of a pair of critical points (this corresponds to a type II Reidemeister move on the xy -diagram) or failure of the Morse-Smale condition. None of these will effect the homology (see section 5.1.2 for explicit isomorphisms). Finally, the only effect of stabilization on the Morse complex is to shift the indices of critical points, but this is counter-acted by the degree shift on homology.

Proposition 5.2 creates a temptation to compare the Morse complex for the function w and the linearized DGA for an augmentation of the front diagram (we need to compare only differentials: the chain spaces are the same by Proposition 5.2). See section 6.1 for a sketch of this approach in the case of the explicitly constructed generating families from section 3.

The difficulty with this approach in general is the lack of a readily apparent augmentation of $\mathbf{A}(\Gamma)$ where Γ is the xy -diagram of ℓ_F itself. However, the analogy between the proofs of Theorem 2.4 and Theorem 2.7 suggests a direct route between the Morse complexes of the individual functions f_t and an augmentation for the xy -diagram obtained from the splash construction. This analogy motivates the proof of the main result of this section.

Theorem 5.3 *If F is generic (as described in the next section) and linear at infinity then there exists a graded augmentation ε for the Chekanov-Eliashberg DGA of ℓ_F such that $\mathcal{GH}_*(F) \cong H_*^\varepsilon(\ell_F)$.*

Outline of proof. In order to compute $H_*(w \geq \delta, w = \delta; \mathbb{Z}_2) \cong H_*(\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}, \{w \leq \delta\}; \mathbb{Z}_2)$ (excision) we replace the pair $(\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}, \{w \leq \delta\})$ with a homotopy equivalent subpair which becomes compact after collapsing $\{w \leq \delta\}$ to a point. On this compact space, we use the fiberwise ascending and descending manifolds of points in S_F to produce a CW-complex structure, and the associated cellular chain complex will be used in the isomorphism.

The set of linearized homology groups when ε is allowed to vary over all augmentations is a Legendrian isotopy invariant. Therefore, we have the freedom to work with the Chekanov-Eliashberg DGA of a different representative of the Legendrian isotopy class of ℓ_F . The representative that is convenient for our purposes is one whose xy -diagram, Γ_F has been made to resemble the front diagram of ℓ_F and then been decorated with splashes as described in section 4.4. An augmentation of $\mathbf{A}(\Gamma_F)$ is defined by examining the bifurcations of the Morse complexes of f_t . The corresponding linearized complex will then appear strikingly similar to a quotient of the previously constructed cellular chain complex by an acyclic subcomplex. Indeed, the last step of the proof is to show that they are quasi-isomorphic.

Remark. A close variant of Theorem 5.3. is proved in [NgTr] and [JTr] for certain classes of 2-component links where the invariants are explicitly calculated.

5.1 Cell decompositions from 1-parameter families

Due to the linear at infinity assumption, for large enough T the spaces $\{w \leq \delta\} \cap \{t \geq T\}$ (resp. $\{w \leq \delta\} \cap \{t \leq -T\}$) is a product $V \times [T, +\infty)$ (resp. $V \times (-\infty, -T]$) where V is a half-space of $\mathbb{R}^N \times \mathbb{R}^N$. Therefore, for the purpose of computing the homology groups in the statement of Theorem 5.3 we can and will henceforth assume our generating family $\{f_t\}$ is defined on a compact interval $t \in [-T, T]$ with $S_F \subset \mathbb{R}^N \times (-T, T)$.

We will need to pair our generating family $\{f_t\}$ with a one parameter family of metrics $\{g_t\}$, $-T \leq t \leq T$ which we assume to be Euclidean outside of some compact set. Then we can consider the fiber-wise negative gradient flow $\Phi_s : \mathbb{R}^N \times [-T, T] \rightarrow \mathbb{R}^N \times [-T, T]$ generated by the vector field $X_{(x,t)} = (-\nabla_{g_t} f_t)_x$. This flow preserves the fibers $\mathbb{R}^N \times t$ and by the linearity assumption is globally defined. For (x, t) , if $\lim_{s \rightarrow \pm\infty} \Phi_s(x, t)$ exists it belongs to $(S_F)_t$. Alternatively, $\Phi_s(x, t)$ eventually follows a line with $\lim_{s \rightarrow \pm\infty} f_t(\Phi_s(x, t)) = \mp\infty$.

Definition 5.1 *Given a subset $B \subset S_F$, let $\mathcal{D}(B) = \{(x, t) \mid \lim_{s \rightarrow -\infty} \Phi_s(x, t) \in B\}$ and $\mathcal{A}(B) = \{(x, t) \mid \lim_{s \rightarrow +\infty} \Phi_s(x, t) \in B\}$ denote the fiber-wise descending and ascending manifolds of B .*

5.1.1 The Morse complex

Our main tool for computations is an extension of a beautiful perspective on Morse theory originally explored by René Thom. On a compact manifold M suppose we are given a single Morse function f with critical points $\text{crit}(f) = \{p_1, \dots, p_n\}$ labeled so that $f(p_1) \geq \dots \geq f(p_n)$. Fix a metric g on M satisfying the Morse-Smale condition:

For all i, j , $\mathcal{D}(p_i)$ and $\mathcal{A}(p_j)$ intersect transversally.

In this case the descending manifolds will form the cells of a CW-complex structure, $M = \coprod \mathcal{D}(p_i)$. The dimension of $\mathcal{D}(p_i)$ is given by the Morse index $\text{ind}(p_i)$.

If $\text{ind}(p_i) = \text{ind}(p_j) + 1$, then the incidence number $\eta(i, j) = [\mathcal{D}(p_i) : \mathcal{D}(p_j)]$ is the number of descending gradient trajectories from p_i to p_j (which comprise $\mathcal{D}(p_i) \cap \mathcal{A}(p_j)$). These trajectories should be counted with signs, but since we consider only modulo 2 homology, we disregard the signs and view $\eta(i, j)$ as integers modulo 2.

To simplify some of our formulas we will make the convention that when $\text{ind}(p_i) \neq \text{ind}(p_j) + 1$, $\eta(i, j) = 0$.

We arrive at the *Morse complex* $(C_*(f), \partial)$ with coefficients in \mathbf{Z}_2 .

$$C_l(f) = \mathbf{Z}_2\{p_i \mid \text{ind}(p_i) = l\}$$

$$\partial : C_l(f) \rightarrow C_{l-1}(f), \partial p_i = \sum_{\text{ind}(p_j)=l-1} \eta(i, j) p_j$$

Note that the differential is strictly upper triangular in the basis $\{p_i\}$. The homology of the Morse complex is isomorphic to the singular homology of M modulo 2.

If we fix two regular values a and $b > a$ of f and restrict the complex to the critical points p_i with $a < f(p_i) < b$, then we get a complex $(C_*(f; a, b), \partial)$ whose homology is $H_*(\{f \leq b\}, \{f \leq a\}; \mathbf{Z}_2)$. The latter will work also in the case when M is non-compact, provided the gradient flow is globally defined.

5.1.2 Bifurcations of the Morse complex

Consider the following two conditions on a C^∞ function f :

- (i) f is Morse;
- (ii) critical values of f are distinct;

These conditions are simultaneously satisfied on a generic (open and dense) subset of the set of all functions.

In the statement of the theorem, the assumption that our family of functions $\{f_t\}$, $-T \leq t \leq T$ is generic means that there will be a finite number of values $-T < t_1 \leq t_2 \leq \dots \leq t_M < T$ where precisely one of (i) and (ii) fails for f_{t_i} .

Further, if f_{t_i} fails to be Morse then it is due to either:

(B) a birth of a pair of critical points with adjacent indices $\lambda + 1$ and λ ,

or

(D) a death of a pair of critical points with adjacent indices $\lambda + 1$ and λ .

Also, when condition (ii) fails it will be through

(TCV) transverse intersection of critical values.

In either case (B) or (D) there will be a unique degenerate critical point p . $\mathcal{D}(p)$ (resp. $\mathcal{A}(p)$) will be a half disk of dimension $\lambda + 1$ (resp. $N - \lambda$ where $N = \dim M$) so the Morse-Smale transversality condition still makes sense in this setting.

Remark. On the front diagram of ℓ_F (B) and (D) points correspond to left and right cusps respectively. The t -values for which two critical points share a common critical value correspond to crossings. Note that the genericity assumption (TCV) guarantees Λ_F is an *embedded* Legendrian submanifold.

We can choose the family of metrics g_t so that the Morse-Smale condition fails at only a finite number of t -values at which conditions (i) and (ii) are both satisfied [L]. The manner in which the Morse-Smale condition fails can be assumed to agree with the description given before Lemma 5.2 below. We use (M - S) to indicate a value of t where the Morse-Smale condition fails for (f_t, g_t) .

Definition 5.2 A t -value such that one of (B), (D), (TCV), or (M - S) occurs will be referred to as a *singular t -value*.

Remark. Throughout the remainder of the proof we will be considering each of these singularity types on a case by case basis. The reader is encouraged to concentrate on one type of singularity at a time in order to avoid becoming bogged down.

Let us examine case by case how the Morse complex changes when we pass a singular t -value and in addition extend the Morse complex to the types of singular pairs (f_t, g_t) described above. We will haphazardly add t 's to our previous notation to indicate which function we are considering. For instance p_i becomes $p_i(t)$, $\eta(i, j)$ becomes $\eta^t(i, j)$, and the differential of the Morse complex becomes ∂_t .

Case of no singularity: If the open interval (a, b) contains no singular t -values then the Morse complex is stable through out. We have $\eta^{t'}(i, j) = \eta^{t''}(i, j)$ for all $a < t', t'' < b$.

In the rest of the cases, for simplicity of notation, we assume that on the interval $(-2, 2)$ there is a lone singularity at $t = 0$ of the desired type. The formulas in Lemma 5.1 and 5.2 come from [L], although the cell-decompositions at singular t -values are not discussed in that reference.

Case (B): Assume the newly born critical points are labeled $p_k(1), p_{k+1}(1)$ with the indices $\lambda + 1, \lambda$. $C(f_1)$ has two more generators than $C(f_{-1})$. To express the relationship between the two complexes it is convenient to fore go our usual labeling convention by listing $\text{crit}(f_{-1}) = \{p_1, \dots, p_{k-1}, p_{k+2}, \dots\}$ (compare with the warning from section 4.1). This notation will be retained in future considerations of (B) t -values.

Differentials are related by the formulas:

$$\begin{aligned}
\partial_1 p_i &= \partial_{-1} p_i + \eta^1(i, k+1) \partial_1 p_k + \eta^1(\partial_{-1} p_i, p_{k+1}) p_k \\
&= \partial_{-1} p_i + \sum_j \eta^1(i, k+1) \eta^1(k, j) p_j + \sum_j \eta^{-1}(i, j) \eta^1(j, k+1) p_k \\
\partial_1 p_k &= p_{k+1} + \sum_{k+1 < j} \eta^1(k, j) p_j \\
\partial_1 p_{k+1} &= \sum_j \eta^1(k, j) \partial_{-1} p_j
\end{aligned}$$

Remark. For $i \neq k, k+1$, unless $\text{ind}(p_i) = \lambda + 2$ or $\lambda + 1$ we have simply $\partial_1 p_i = \partial_{-1} p_i$. In the use of $\eta^1(\partial_{-1} p_i, p_{k+1})$ we extend $\eta^1(\cdot, p_{k+1})$ as a linear form.

Let us record some observations.

Lemma 5.1 (i) $C(f_{-1}) \cong C(f_1)/\mathbf{Z}_2\{p_k \mapsto \partial_1 p_k\}$, $p_i \mapsto [p_i]$ (isomorphism of complexes).

(ii) In $C(f_1)/\mathbf{Z}_2\{p_k \mapsto \partial_1 p_k\}$,

$$[p_{k+1}] = \sum_{j \neq k+1} \eta^1(k, j) [p_j]$$

(iii) The map $A : C(f_{-1}) \oplus \mathbf{Z}_2\{p_k, p_{k+1}\} \rightarrow C(f_1)$, $A(p_i) = p_i + \eta^1(i, k+1) p_k$ for $i \neq k, k+1$, $A(p_k) = p_k$, $A(p_{k+1}) = p_{k+1} + \sum_{j \neq k+1} \eta^1(k, j) p_j$ is an isomorphism of complexes where $C(f_{-1}) \oplus \mathbf{Z}_2\{p_k, p_{k+1}\}$ is as a complex the split extension of $C(f_{-1})$ by $p_k \mapsto p_{k+1}$.

The descending manifolds with respect to (f_0, g_0) will give a CW-complex structure and we define the ‘Morse complex’ $C(f_0)$ to be the cellular chain complex. The only abnormality is that the descending manifold of the degenerate critical point is a half disc so contributes 2 cells. If we label these cells as p_k, p_{k+1} then $C(f_0)$ will be identical to $C(f_1)$.

Case (D): The situation is symmetric to the case (B).

Case (TCV): The only thing that changes is the way we should label critical points. If the crossing occurs between critical points labeled p_k, p_{k+1} then the matrix $(\eta^{-1}(i, j))$ agrees with $(\eta^1(i, j))$ after conjugating by the permutation matrix of the transposition $(k, k+1)$. The Morse complex at $t = 0$ makes sense as usual. By convention we let the labeling of critical points at $t = 0$ agree with the labeling when $t < 0$.

Case (M-S): The generic way that this will happen is that at some point a gradient flow line will connect two critical points $p_k, p_l, k < l$ with the same

index, λ . This prevents a CW-complex structure by descending manifolds since the closure of $\mathcal{D}(p_k)$ will intersect $\mathcal{D}(p_l)$. To rectify this problem we divide the cell $\mathcal{D}(p_l)$ into three cells $\mathcal{D}(p_l)^-, \mathcal{D}(p_l)^+, \mathcal{D}(p_l)^0$ where

$$\begin{aligned}\mathcal{D}(p_l)^0 &= \overline{\mathcal{D}(p_k)} \cap \mathcal{D}(p_l) \\ \mathcal{D}(p_l)^- &= (\overline{\mathcal{D}(p_k(t), t < 0)} \cap \mathcal{D}(p_l)) - \mathcal{D}(p_l)^0 \\ \mathcal{D}(p_l)^+ &= (\overline{\mathcal{D}(p_k(t), t > 0)} \cap \mathcal{D}(p_l)) - \mathcal{D}(p_l)^0\end{aligned}$$

Here $\mathcal{D}(p_l)^0$ is a $(\lambda - 1)$ -cell while both $\mathcal{D}(p_l)^-$ and $\mathcal{D}(p_l)^+$ are λ -cells.

In the Morse complex for $C(f_0)$, which is defined to be the cellular chain complex obtained from this decomposition, we use p_l^0, p_l^-, p_l^+ to denote the three pieces of $\mathcal{D}(p_l)$. Relationships between the complexes $C(f_i), i = -1, 0, 1$ are

$$\begin{aligned}\partial_{-1}p_l &= \partial_1p_l = \partial_0p_l^- + \partial_0p_l^+ \\ \partial_0p_l^+ &= p_l^0 + x, \partial_0p_l^- = p_l^0 + y\end{aligned}$$

where $x, y \in \text{span}\{p_i | i \neq l\}$,

$$\partial_0p_k = \partial_{-1}p_k + \partial_0p_l^- = \partial_1p_k + \partial_0p_l^+,$$

and if $\text{ind}(p_i) = \lambda + 1$, then

$$\partial_0p_i = \partial_{-1}p_i + \eta(i, k)p_l^- = \partial_1p_i + \eta(i, k)p_l^+$$

where all appearances of p_l in the last two formulas should be replaced by $p_l^- + p_l^+$.

We will use some simple consequences.

Lemma 5.2 *There are isomorphisms of complexes:*

- (i) $C(f_{-1}) \cong C(f_0)/\mathbf{Z}_2\{p_l^- \mapsto \partial_0p_l^-\}$ given by $p_i \mapsto [p_i], i \neq l$ and $p_l \mapsto [p_l^+]$;
- (ii) $C(f_1) \cong C(f_0)/\mathbf{Z}_2\{p_l^+ \mapsto \partial_0p_l^+\}$ given by $p_i \mapsto [p_i], i \neq l$ and $p_l \mapsto [p_l^-]$;
- (iii) $A : C(f_{-1}) \xrightarrow{\cong} C(f_1)$ given by $A(p_i) = p_i, i \neq k$ and $A(p_k) = p_k + p_l$.

5.1.3 A CW-complex structure for $M \times [-T, T]$

Let M be a compact smooth manifold with a generic family of functions and metrics $(f_t, g_t), -T \leq t \leq T$. Let $-T = t_0 < t_1 < \dots < t_M = T$ denote a partitioning of the interval which includes all singular t -values and one

non-singular t -value between any two of the singular values. As described in Section 5.1.2, for each t_m we have a CW-decomposition of $M \times \{t_m\}$ whose cellular chain complex is the Morse complex $C(f_{t_m})$. We shorten the notation for cells to $p_i(m) = \mathcal{D}(p_i(t_m))$. On an interval (t_{m-1}, t_m) the Morse complex $C(f_t)$ is stable and we can form cells $P_i(m) = \mathcal{D}(p_i(t), t_{m-1} < t < t_m)$ of dimension $\text{ind}(p_i) + 1$.

The following theorem incorporates the bifurcation data to give formulas for the differential.

Theorem 5.4 *The decomposition*

$$M \times [-T, T] = (\coprod_m (\coprod_i p_i(m))) \coprod (\coprod_m (\coprod_i P_i(m)))$$

is compatible with a CW-complex structure. The cellular chain complex as a vector space is a direct sum $(\oplus_m A(m)) \oplus (\oplus_m B(m))$ where $A(m)$ is spanned by the cells belonging to $M \times \{t_m\}$, and $B(m)$ is spanned by cells belonging to $M \times (t_{m-1}, t_m)$.

- (a) For each m , $A(m) = C(f_{t_m})$ is a sub-complex.
- (b) For each m , the differential on $B(m)$ is as a sum of three parts,

$$\begin{aligned} \partial_{B(m), B(m)} : B(m) &\rightarrow B(m) \\ \partial_{B(m), A(m-1)} : B(m) &\rightarrow A(m-1) \\ \partial_{B(m), A(m)} : B(m) &\rightarrow A(m) \end{aligned}$$

The first part acts according to

$$\partial_{B(m), B(m)} P_i(m) = \sum_j \eta^t(i, j) P_j(m),$$

the formula being independent of the choice of t , $t_{m-1} < t < t_m$.

The part $\partial_{B(m), A(m-1)} : B(m) \rightarrow A(m-1)$ (resp. $\partial_{B(m), A(m)} : B(m) \rightarrow A(m)$) is defined depending on the nature of the singularity at t_{m-1} (resp. t_m).

In the case, when there is no singularity, $\partial_{B(m), A(m-1)} P_i(m) = p_i(m-1)$ (resp. $\partial_{B(m), A(m)} P_i(m) = p_i(m)$).

In the case (B), $\partial_{B(m), A(m-1)} P_i(m) = p_i(m-1)$ (resp. $\partial_{B(m), A(m)} P_i(m) = p_i(m) + \eta^{t_m}(i, k+1) p_k(m)$). (The convention that critical points on the interval (t_{m-1}, t_m) are labeled to omit p_k and p_{k+1} is used here.)

In the case (D), $\partial_{B(m), A(m-1)} P_i(m) = p_i(m) + \eta^{t_{m-1}}(i, k+1) p_k(m)$ under the convention that critical points on the interval (t_{m-1}, t_m) are labeled to omit p_k and p_{k+1} (resp. $\partial_{B(m), A(m)} P_i(m) = p_i(m)$).

In the case (TCV), $\partial_{B(m),A(m-1)}P_i(m) = p_{\sigma(i)}(m-1)$ where $\sigma = (k, k+1)$ (resp. $\partial_{B(m),A(m)}P_i(m) = p_i(m)$). (The asymmetry is due to the labeling convention.)

In the case (M-S):

if $i \neq k, l$, $\partial_{B(m),A(m-1)}P_i(m) = p_i(m-1)$, (resp. $\partial_{B(m),A(m)}P_i(m) = p_i(m)$);
if $i = k$, $\partial_{B(m),A(m-1)}P_k(m) = p_k(m-1) + p_l^+(m-1)$, (resp. $\partial_{B(m),A(m)}P_k(m) = p_k(m) + p_l^-(m)$);
if $i = l$, $\partial_{B(m),A(m-1)}P_l(m) = p_l^-(m-1) + p_l^+(m-1)$ (resp. $\partial_{B(m),A(m)}P_l(m) = p_l^-(m) + p_l^+(m)$).

Remark. In the case we are primarily interested in $M = \mathbb{R}^N$ is non-compact. Then the above complex computes $H_*(\mathbb{R}^N, F \leq -C)$ where C is large enough so that $(-\infty, -C)$ does not contain any critical values. The cells P_i, p_i now give a decomposition of $\mathcal{D}(S_F)/\{F \leq -C\}$ instead of all of $\mathbb{R}^N \times [-T, T]$.

5.1.4 Cellular chain complex on the fiber product

We return now to the specialized case of a linear at infinity family $f_t : \mathbb{R}^N \rightarrow \mathbb{R}$ with metrics g_t chosen as above. Let w_t denote the family corresponding to difference function $w : \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$, so that $w_t(x, y) = f_t(x) - f_t(y)$.

Theorem 5.2 does not apply directly to the pair $(w_t, g_t \oplus g_t)$. For instance, when the Morse-Smale condition is violated by a pair of critical points $p_k(t), p_l(t)$ of f_t it will also be violated by a much larger number of the critical points for w_t . However, we can avoid applying the theorem to $(w_t, g_t \oplus g_t)$.

Note that the descending manifolds of $-f_t$ are simply the ascending manifolds of f_t . As in the previous section we get cell decompositions for both $\mathcal{D}(S_F)$ and $\mathcal{A}(S_F)(= \mathcal{D}(S_{-F}))$. The notation for the ascending manifold cells will use q 's instead of p 's, but will otherwise be identical. We use $A'(m)$ and $B'(m)$ for the vector spaces spanned by $q_i(m)$ and $Q_i(m)$ respectively. Note that for a non-singular t -value (and hence for any t -value except for those where (M-S) fails) the Morse complexes $C(f_t)$ and $C(-f_t)$ are dual: $\partial_t q_i = \sum_j \eta^t(j, i) q_j$. At birth-death t -values the role of p_k (resp. p_{k+1}) is played by q_{k+1} (resp. q_k). Similarly, at t -values where the Morse-Smale condition fails the role of p_k (resp. p_l) is played by q_l (resp. q_k).

There are product cells $p_i(m) \times q_j(m)$ which will have dimension $N + \text{ind}(p_i) - \text{ind}(p_j)$, and since the cells $P_i(m)$ and $Q_j(m)$ are fibered over the

interval as $B^{\text{ind}(p_i)} \times B^1$ and $B^{N-\text{ind}(p_j)} \times B^1$ the fiber product $P_i(m) * Q_j(m)$ is a $1 + N + \text{ind}(p_i) - \text{ind}(p_j)$ cell. Put together, we get a cell decomposition of $\mathcal{D}(S_F) * \mathcal{A}(S_F) = \mathcal{D}(S_w)$. A CW-decomposition of $\mathcal{D}(S_w)/\{w \leq \delta\}$ arises from the above cell decomposition. This is because on the cells in the descending (resp. ascending) manifold F decreases (resp. increases) as we move away from the critical set. Therefore, on the (fiber) product cells w decreases as we move away from the critical points in the center, and the portion of a cell with $w \leq \delta$ will itself be a cell when non-empty. The collapsing of $\{w \leq \delta\}$ results in a compact space and adds an additional 0-cell.

The cellular chain complex (\mathbf{C}, ∂) relative to the 0-cell $\{w \leq \delta\}$ as a vector space can be realized as a quotient of the direct sum $(\oplus_m A(m) \otimes A'(m)) \oplus (\oplus_m B(m) \otimes B'(m))$ by the subspace generated by

- (i) $p_i(m) \otimes q_j(m)$ and $P_i(m) \otimes Q_j(m)$ when $i \geq j$ (including some additional cells with superscripts $+$, $-$, 0 when the Morse-Smale condition fails).
- (ii) $p_k(m) \otimes p_{k+1}(m)$ if t_m is a (B) , (D) , or (TCV) singular t -value and $p_k(m)$, $p_{k+1}(m)$ are the offending critical points.

We denote the images of $A(m) \otimes A'(m)$ and $B(m) \otimes B'(m)$ in the quotient as $\mathbf{A}(m)$ and $\mathbf{B}(m)$, but we will not use a new notation to distinguish between a generator and its coset.

Our identification with the cellular chain complex is by $p_i(m) \times q_j(m) \leftrightarrow p_i(m) \otimes q_j(m)$ and $P_i(m) * Q_j(m) \leftrightarrow P_i(m) \otimes Q_j(m)$.

The $\mathbf{A}(m)$ are subcomplexes with the usual tensor product differential,

$$\partial_{A(m)} \otimes \mathbf{1}_{A'(m)} + \mathbf{1}_{A(m)} \otimes \partial_{A'(m)}$$

The differential on $\mathbf{B}(m)$ is a sum

$$\begin{aligned} \partial_{B(m), B(m)} \otimes \mathbf{1}_{B'(m)} + \mathbf{1}_{B(m)} \otimes \partial_{B'(m), B'(m)} & : \mathbf{B}(m) \rightarrow \mathbf{B}(m) \\ + \partial_{B(m), A(m-1)} \otimes \partial_{B'(m), A'(m-1)} & : \mathbf{B}(m) \rightarrow \mathbf{A}(m-1) \\ + \partial_{B(m), A(m)} \otimes \partial_{B'(m), A'(m)} & : \mathbf{B}(m) \rightarrow \mathbf{A}(m) \end{aligned}$$

All of the above maps are well defined on the quotient.

5.1.5 A quasi-isomorphic quotient

In this section we take a quotient of the complex constructed in the previous section by an acyclic sub-complex \mathbf{E} . Motivated by linearized complexes

coming from the Chekanov-Eliashberg DGA our goal is to make the number of generators in $\mathbf{A}(m)$ and $\mathbf{B}(m)$ roughly the same. We then provide suggestive notation for a basis of the quotient complex and record the formula for the differential.

The sub-complex \mathbf{E} is the direct sum of sub-complexes of the $\mathbf{A}(m)$. The intersection $\mathbf{E}(m) := \mathbf{E} \cap \mathbf{A}(m)$ is defined depending on the type of singularity at t_m .

Case (B). Let $\mathbf{F}(m)$ be spanned by $\{p_k(m) \otimes q_j(m), p_i(m) \otimes q_{k+1}(m) \mid k+1 < j \text{ and } i < k\}$. It follows from Lemma 5.1 (since coefficient of p_{k+1} in ∂p_k is 1, the coefficient of q_k in ∂q_{k+1} is 1, and $p_k(m) \otimes q_{k+1}(m) = 0$ in $\mathbf{A}(m)$) that ∂ maps $\mathbf{F}(m)$ isomorphically onto its image so that $\mathbf{E}(m) := \text{span } \mathbf{F}(m) \cup \partial \mathbf{F}(m)$ is an acyclic subcomplex.

Case (M-S). Let $\mathbf{F}(m) = \text{span}\{p_l^- \otimes q_j, p_i \otimes q_k^- \mid l < j \text{ and } i < k\}$. It follows from Lemma 5.2 (since coefficient of p_l^0 in ∂p_l^- is 1, coefficient of q_k^0 in ∂q_k^- is 1, and $p_l \otimes q_k = 0$ in $\mathbf{A}(m)$ regardless of superscripts) that ∂ maps $\mathbf{F}(m)$ isomorphically onto its image so that $\mathbf{E}(m) = \text{span } \mathbf{F}(m) \cup \partial \mathbf{F}(m)$ is an acyclic subcomplex.

All other cases: $\mathbf{E}(m) = \{0\}$.

We now use $\overline{\mathbf{C}} = (\oplus_m \overline{\mathbf{A}}(m)) \oplus (\oplus_m \overline{\mathbf{B}}(m))$ for the quotient complex \mathbf{C}/\mathbf{E} with inherited direct sum decomposition. We use $[x]$ to denote the coset of an element $x \in \mathbf{C}$. Since \mathbf{E} is acyclic the homology groups of $\overline{\mathbf{C}}$ compute $H_*(\mathcal{D}(S_w), w \leq \delta; \mathbb{Z}_2) \cong H_*(\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}, w \leq \delta; \mathbb{Z}_2)$.

To set up the desired isomorphism of homology groups, at this stage we single out a specific basis for $\overline{\mathbf{C}}$. It will be a union of bases $\{\mathbf{r}_{m;i,j}^+\}$ for $\overline{\mathbf{A}}(m)$ and $\{\mathbf{r}_{m;i,j}^-\}$ for $\overline{\mathbf{B}}(m)$. (The reader may notice a similarity between this notation and the notation in Section 4.1. This is done in purpose: the similarly denoted generators will be put into correspondence with each other on the final stage of the proof of Theorem 5.3.)

Not surprisingly the definition of $\{\mathbf{r}_{m;i,j}^+\}$ depends on the type of singularity at t_m .

Case (B): $\mathbf{r}_{m;i,j}^+ = [p_i(m) \otimes q_j(m)] \in \overline{\mathbf{A}}(m), i < j, \{i, j\} \cap \{k, k+1\} = \emptyset$. Note that from Lemma 3.1

$$\begin{aligned} [p_{k+1}(m) \otimes q_j(m)] &= \sum_l \eta^{t_{m+1}}(k, l) \mathbf{r}_{m;l,j}^+, \\ [p_i(m) \otimes q_k(m)] &= \sum_l \eta^{t_{m+1}}(l, k+1) \mathbf{r}_{m;i,l}^+. \end{aligned}$$

Case (M-S) :

$$\begin{aligned}\mathfrak{r}_{m;i,j}^+ &= [p_i(m) \otimes q_j(m)], i < j, i \neq l, j \neq k, \\ \mathfrak{r}_{m;l,j}^+ &= [p_l^+(m) \otimes q_j(m)], \mathfrak{r}_{m;i,k}^+ = [p_i(m) \times q_k^+(m)].\end{aligned}$$

All other cases: $\mathfrak{r}_{m;i,j}^+ = p_i(m) \otimes q_j(m) \in \overline{\mathbf{A}}(m), i < j$ where if two critical values p_k, p_{k+1} intersect at t_m , $x_{m;k,k+1}^+$ is not defined (This is because w is non-positive on $\mathcal{D}(p_k) \times \mathcal{A}(p_{k+1})$).

Finally, for each m and $i < j$ define $\mathfrak{r}_{m;i,j}^- = P_i(m) \otimes Q_j(m)$.

Lemma 5.3 *The elements $\mathfrak{r}_{m;i,j}^-, \mathfrak{r}_{m;i,j}^+$ as defined above form a basis for $\overline{\mathbf{C}}$.*

To conclude this section we record formulas for the differential of $\overline{\mathbf{C}}$ with respect to the basis from the lemma.

On the subcomplexes $\overline{\mathbf{A}}(m)$ if t_m is a non-singular t -value then

$$\partial \mathfrak{r}_{m;i,j}^+ = \sum_{i < l < j} \eta^{t_m}(i, l) \mathfrak{r}_{m;l,j}^+ + \sum_{i < l < j} \eta^{t_m}(l, j) \mathfrak{r}_{m;i,l}^+$$

If t_m is a singular t -value then the differential for $\overline{\mathbf{A}}(m)$ agrees with the differential for $\overline{\mathbf{A}}(m-1)$ except that the term $\mathfrak{r}_{m;k,k+1}^+$ does not exist if the singular value is a crossing or a right cusp. (t_{m-1} was assumed to be non-singular if t_m is singular.) This is a consequence of Lemmas 5.1 and 5.2 and in fact **E** was chosen specifically so this would be the case.

$$\partial \mathfrak{r}_{m;i,j}^- = \widehat{\mathfrak{r}}_{m-1;i,j}^+ + \sum_{i < l < j} \eta^t(i, l) \mathfrak{r}_{m;l,j}^- + \sum_{i < l < j} \eta^t(l, j) \mathfrak{r}_{m;i,l}^- + \mathfrak{r}_{m;i,j}^+$$

where the term $\widehat{\mathfrak{r}}_{m-1;i,j}^+ \in \overline{\mathbf{A}}(m-1)$ depends on the type of singularity at t_{m-1} .

Case of no singularity:

$$\widehat{\mathfrak{r}}_{m-1;i,j}^+ = \mathfrak{r}_{m-1;i,j}^+$$

Case (B): Assuming $\{i, j\} \cap \{k, k+1\} = \emptyset$, $\widehat{\mathfrak{r}}_{m-1;i,j}^+ = \mathfrak{r}_{m-1;i,j}^+$. Also,

$$\begin{aligned}\widehat{\mathfrak{r}}_{m-1;k,j}^+ &= \widehat{\mathfrak{r}}_{m-1;i,k+1}^+ = \widehat{\mathfrak{r}}_{m-1;k,k+1}^+ = 0 \\ \widehat{\mathfrak{r}}_{m-1;k+1,j}^+ &= \sum_l \eta^m(k, l) \mathfrak{r}_{m-1;l,j}^+ \\ \widehat{\mathfrak{r}}_{m-1;i,k}^+ &= \sum_l \eta^m(l, k+1) \mathfrak{r}_{m-1;i,l}^+\end{aligned}$$

Case (D):

$$\widehat{\mathfrak{x}}_{m-1;i,j}^+ = \mathfrak{x}_{m-1;i,j}^+ + \eta^{m-1}(i, k+1)\mathfrak{x}_{m-1;k,j}^+ + \eta^{m-1}(k, j)\mathfrak{x}_{m-1;i,k+1}^+$$

Case (TCV):

$$\widehat{\mathfrak{x}}_{m-1;i,j}^+ = \mathfrak{x}_{m-1;\sigma(i),\sigma(j)}^+$$

where σ is the transposition $(k, k+1)$.

Case (M-S): Assuming $i \neq k, j \neq l, \widehat{\mathfrak{x}}_{m-1;i,j}^+ = \mathfrak{x}_{m-1;i,j}^+$. Also,

$$\begin{aligned}\widehat{\mathfrak{x}}_{m-1;k,j}^+ &= \mathfrak{x}_{m-1;k,j}^+ + \mathfrak{x}_{m-1;l,j}^+ \\ \widehat{\mathfrak{x}}_{m-1;i,l}^+ &= \mathfrak{x}_{m-1;i,l}^+ + \mathfrak{x}_{m-1;i,k}^+ \\ \widehat{\mathfrak{x}}_{m-1;k,l}^+ &= \mathfrak{x}_{m-1;k,l}^+\end{aligned}$$

Remark. The fact that the term of $\partial\mathfrak{x}_{m;i,j}^-$ belonging to $\overline{\mathbf{A}}(m)$ is always $\mathfrak{x}_{m;i,j}^+$ follows from Lemma 3.1 and 3.2 and the choice of \mathbf{E} . In the coefficient $\eta^t(i, l)$ (resp. $\eta^t(l, j)$) of $\mathfrak{x}_{m;l,j}^-$ (resp. $\mathfrak{x}_{m;i,l}^-$) t needs to satisfies $t_{m-1} < t < t_m$, and is independent of the choice since the Morse complex is stable on the interval (t_{m-1}, t_m) . Again, if there is a crossing or right cusp at t_m then $\mathfrak{x}_{m;k,k+1}^+ = 0$.

5.2 Construction of the augmentation

Instead of using the Chekanov-Eliashberg DGA as defined by the xy -projection of ℓ_F itself, we first apply Ng's resolution procedure to the front projection of ℓ_F and add a certain number of splashes (see Section 4.4).

Specifically, recall that we have made a subdivision $-T \leq t_0 < t_1 < \dots < t_M \leq T$ (T should be large enough so that f_t is linear outside this interval) so that

- (i) every singular t -value is a t_i and
- (ii) the sequence of $\{t_m\}$ alternates between singular and non-singular t -values.

For Λ'_F we add one splash for each of the t_m . If t_m is a non-singular t -value the splash is contained in a small interval about t_m . If t_m is a singular t -value then place the splash in an interval directly to the left of the singular point. This is not so important for (M-S) singularities since they are not reflected by the front diagram of a knot. What is important is that at crossings or cusps the splash is placed directly to the left of the singularity. We get a

related partitioning $-T \leq s_0 < s_1 < \dots < s_M \leq T$ where s_m is a point chosen from the interval containing the m -th splashing.

Denote the xy -projection of the resulting knot by Γ_F .

5.2.1 An augmentation for $\mathbf{A}(\Gamma_F)$

Refer to section 4.4 for notation of generators and formulas for the differential of the corresponding Chekanov-Eliashberg DGA. We now construct an augmentation $\varepsilon: \mathbf{A}(\Gamma_F) \rightarrow \mathbb{Z}_2$. It is important that although we use for our augmentation the same notation ε as in Section 4.5, the construction is different. No wonder: in Section 4.5 we used a normal ruling, and here we use a generating family.

On the generators y_m, z_m coming from crossing and right cusp inserts set

$$\varepsilon(y_m) = 0, \varepsilon(z_m) = 0.$$

We define

$$\varepsilon(x_{m;ij}^+) = \eta^{s_m}(i, j)$$

so that $\varepsilon(X_m^+)$ is the matrix of the differential in $C(f_{s_m})$ with respect to the basis $\{p_i(s_m)\}$. Since $\partial X_m^+ = (X_m^+)^2$, it follows that $\varepsilon(\partial X_m^+) = 0$. Since two critical points whose critical values meet at a crossing (resp. right cusp) cannot be (resp. must be) joined by a gradient trajectory we see also that $\varepsilon(\partial y_m) = \varepsilon(x_{m-1;k,k+1}^+) = 0$ (resp. $\varepsilon(\partial z_m) = \varepsilon(x_{m-1;k,k+1}^+) + 1 = 0$).

In general, $\partial X_m^- = X_m^+(I + X_m^-) + (I + X_m^-)\tilde{X}_{m-1}^+$ (compare with the similar formulas in Section 4.2) where the definition of \tilde{X}_{m-1}^+ depends on the type of insert appearing between the $(m-1)$ -th and m -th laminated zones. In all cases $\varepsilon(\tilde{X}_{m-1}^+)$ is already specified, and turns out to be the matrix of the differential in a complex closely related to $C(f_{s_{m-1}})$. The condition $\varepsilon(X_m^-) = 0$ will be satisfied provided $I + \varepsilon(X_m^-)$ is the matrix of an isomorphism between the two relevant complexes. Such isomorphisms are provided in Section 5.2.2. Please note that due to our (backwards) conventions for matrices of linear maps, compositions of linear maps correspond to matrix products in the reverse order.

$\varepsilon(X_m^-)$ is defined depending on the type of singularity at t_{m-1} .

Case of no singularity. In this case $\varepsilon(\tilde{X}_{m-1}^+) = \varepsilon(X_{m-1}^+) = \varepsilon(X_m^+)$ and we define $\varepsilon(X_m^-) = 0$.

Case (B). In this case $\varepsilon(\tilde{X}_{m-1}^+)$ is the matrix of the split extension of $C(f_{s_{m-1}})$ by $\mathbb{Z}_2\{p_k \mapsto p_{k+1}\}$, provided we use our usual convention for labeling

critical points. We define $\varepsilon(X_m^-)$ in such a way that $I + \varepsilon(X_m^-)$ is the matrix of the isomorphism A from Lemma 5.1:

$$\begin{aligned}\varepsilon(x_{m;i,k}^-) &= \eta^{s_m}(i, k+1), \varepsilon(x_{m;k+1,j}^-) = \eta^{s_m}(k, j) \\ \varepsilon(x_{m;i,j}^-) &= 0 \text{ when } i \neq k+1 \text{ and } j \neq k.\end{aligned}$$

Case (D). In this case, $\varepsilon(\tilde{x}_{m-1;i,j}^+) = \eta^{s_{m-1}}(i, j) + \eta^{s_{m-1}}(i, k+1)\eta^{s_{m-1}}(k, j)$, and we observe that $\varepsilon(\tilde{X}_{m-1}^+)$ is the matrix of the differential in the quotient $C(f_{s_{m-1}})/\mathbf{Z}_2\{p_k \mapsto \partial p_k\}$ with respect to the basis $\{[p_1], \dots, [p_{k-1}], [p_{k+2}], \dots\}$. According to the Lemma 5.1 this matrix will be the same as $\varepsilon(X_m^+)$, so we can define $\varepsilon(X_m^-) = 0$.

Case (TCV). In this case, $\tilde{X}_{m-1}^+ = B\hat{X}_{m-1}^+B^{-1}$ where $B = P_{(k,k+1)} + y_mE_{k+1,k+1}$, and $B^{-1} = P_{(k,k+1)} + y_mE_{k,k}$. Here $P_{(k,k+1)}$ is the permutation matrix for the transposition $(k, k+1)$, and $E_{k,k}, E_{k+1,k+1}$ are matrices with a single non-zero entry. \hat{X}_{m-1}^+ is simply X_{m-1}^+ with the entry $x_{m-1;k,k+1}^+$ replaced by 0.

Since $\eta^{s_{m-1}}(k, k+1) = \eta^{s_m}(k, k+1) = 0$ necessarily due to the crossing,

$$\varepsilon(B\hat{X}_{m-1}^+B^{-1}) = P_{(k,k+1)}\varepsilon(X_{m-1}^+)P_{(k,k+1)} = \varepsilon(X_m^+).$$

The last equality follows since the complexes $C(f_{s_{m-1}})$ and $C(f_{s_m})$ differ only by the ordering of generators. We define $\varepsilon(X_m^-) = 0$.

Case (M-S). In this case, $\tilde{X}_{m-1}^+ = X_{m-1}^+$. We define $\varepsilon(X_m^-)$ so that $I + \varepsilon(X_m^-)$ is the matrix of the isomorphism $A : C(f_{s_{m-1}}) \xrightarrow{\cong} C(f_{s_m})$ from Lemma 5.2.,

$$\varepsilon(x_{m;k,l}^-) = 1, \varepsilon(x_{m;i,j}^-) = 0, (i, j) \neq (k, l)$$

In the discussion surrounding the definition of ε we have proved

Lemma 5.4 ε is an augmentation of $\mathbf{A}(\Gamma_F)$.

5.2.2 The differential in the linearized complex

We record formulas for the differential in the associated linearized complex $(\mathbf{A}^\varepsilon, d^\varepsilon)$.

$$\begin{aligned}d^\varepsilon y_m &= x_{m-1;k,k+1}^+, d^\varepsilon z_m = x_{m-1;k,k+1}^+, \\ d^\varepsilon x_{m;i,j}^+ &= \sum_l \eta^{s_m}(i, l)x_{m;l,j}^+ + \sum_l \eta^{s_m}(l, j)x_{m;i,l}^+.\end{aligned}$$

The formulas for $d^\varepsilon x_{m;i,j}^-$ depends on the type of singularity at t_{m-1} .

Case of no singularity:

$$d^\varepsilon x_{m;i,j}^- = x_{m-1;i,j}^+ + x_{m;i,j}^+ + \sum_l \eta^{s_m}(i, l) x_{m;l,j}^- + \sum_l \eta^{s_m-1}(l, j) x_{m;i,l}^-$$

In this case $\eta^{s_m}(i, j) = \eta^{s_m-1}(i, j) = \eta^t(i, j)$ for any $t \in [s_{m-1}, s_m]$.

Case (B): Assuming $\{i, j\} \cap \{k, k+1\} = \emptyset$,

$$\begin{aligned} d^\varepsilon x_{m;i,j}^- &= x_{m;i,j}^+ + \eta^{s_m}(k, j) x_{m;i,k+1}^+ + x_{m-1;i,j}^+ + {}^\circ x^- \\ d^\varepsilon x_{m;k,j}^- &= x_{m;k,j}^+ + \eta^{s_m}(k, j) x_{m;k,k+1}^+ + {}^\circ x^- \\ d^\varepsilon x_{m;k+1,j}^- &= x_{m;k+1,j}^+ + \sum_l \eta^{s_m}(k, l) x_{m-1;l,j}^+ + {}^\circ x^- \\ d^\varepsilon x_{m;i,k}^- &= x_{m;i,k}^+ + \sum_l \eta^{s_m}(l, k+1) x_{m;i,l}^+ + {}^\circ x^- \\ d^\varepsilon x_{m;i,k+1}^- &= x_{m;i,k+1}^+ + {}^\circ x^- \\ d^\varepsilon x_{m;k,k+1}^- &= x_{m;k,k+1}^+ + {}^\circ x^- \end{aligned}$$

where ${}^\circ x^- \in \text{span}\{x_{m;i,j}^-\}$ is a term which will be irrelevant to our argument.

Case (D):

$$\begin{aligned} d^\varepsilon x_{m;i,j}^- &= x_{m;i,j}^+ + \eta^{s_m-1}(i, k+1) x_{m-1;k,j}^+ + \eta^{s_m-1}(k, j) x_{m-1;i,k+1}^+ \\ &\quad + \sum_{i < l < j} \eta^{s_m}(i, l) x_{m;l,j}^- + \sum_{i < l < j} \eta^{s_m}(l, j) x_{m;i,l}^- + {}^\circ z \end{aligned}$$

where ${}^\circ z \in \text{span}\{z_m\}$ will be irrelevant to the argument.

Remark. The term $\sum_{i < l < j} \eta^{s_m}(i, l) x_{m;l,j}^-$ comes from linearizing $X_m^+(I + X_m^-)$ while the similar term $\sum_{i < l < j} \eta^{s_m}(l, j) x_{m;i,l}^-$ comes from linearizing $(I + X_m^-) \tilde{X}_{m-1}^+$. The reason that they both use the Morse complex at s_m is that, as mentioned above, $\varepsilon(X_m^+) = \varepsilon(\tilde{X}_{m-1}^+)$.

Case (TCV):

$$\begin{aligned} \partial x_{m;i,j}^- &= x_{m;i,j}^+ + x_{m-1;\sigma(i),\sigma(j)}^+ + \sum_{i < l < j} \eta^{s_m}(i, l) x_{m;l,j}^- \\ &\quad + \sum_{i < l < j} \eta^{s_m}(l, j) x_{m;i,l}^- + {}^\circ y \end{aligned}$$

where ${}^\circ y \in \text{span}\{y_m\}$ will be irrelevant to the argument.

Case (M-S). Assuming $i \neq k, j \neq l$,

$$\begin{aligned} d^\varepsilon x_{m;i,j}^- &= x_{m;i,j}^+ + x_{m-1;i,j}^+ + {}^\circ x^- \\ d^\varepsilon x_{m;k,j}^- &= x_{m;k,j}^+ + x_{m-1;k,j}^+ + x_{m-1;l,j}^+ + {}^\circ x^- \\ d^\varepsilon x_{m;i,l}^- &= x_{m;i,l}^+ + x_{m-1;i,l}^+ + x_{m;i,k}^+ + {}^\circ x^- \\ d^\varepsilon x_{m;k,l}^- &= x_{m;k,l}^+ + x_{m-1;k,l}^+ + {}^\circ x^- \end{aligned}$$

where ${}^\circ x^- \in \text{span}\{x_{m;i,j}^-\}$ is a term which will be irrelevant to our argument.

5.2.3 Another quasi-isomorphic quotient

The extra generators y_m, z_m from crossings and right cusps generate an acyclic subcomplex which has the basis $\{y_m, x_{m-1;k,k+1}^+\} \cup \{z_m, x_{m-1;k,k+1}^+\}$ where m ranges over all values such that the m -th insert is a crossing or right cusp. We will work with the quotient and retain our previous notation so that an element now denotes its coset in the quotient. The remaining basis elements are then exactly the same as the basis elements for the quotient of the cellular chain complex constructed in Section 5.1.5.

5.3 The isomorphism between homology groups

For the remainder of the proof we use the correspondence $\mathfrak{x}_{m;i,j}^\pm \leftrightarrow x_{m;i,j}^\pm$ to view the two complexes as being defined on the same vector space $\overline{\mathbf{C}}$ (with a grading shift by $N + 1$). We denote the differential inherited from the linearized complex by d^ε and the differential inherited from the cellular chain complex as ∂ . We use the direct sum decomposition $\overline{\mathbf{C}} = (\oplus_m \overline{\mathbf{A}}(m)) \oplus (\oplus_m \overline{\mathbf{B}}(m))$ from section 5.1.5. The proof of Theorem 5.3 will be completed by the following lemma.

Lemma 5.5 *The homology groups of $(\overline{\mathbf{C}}, d^\varepsilon)$ and $(\overline{\mathbf{C}}, \partial)$ are isomorphic.*

Proof of Lemma Observe, that d^ε and ∂ are identical on $\overline{\mathbf{A}}(m)$ for all m , and also on $\overline{\mathbf{B}}(m)$ in all cases except when t_{m-1} is a singular t -value of type (B) or $(M-S)$.

Let

$$D_1 = \text{span } \overline{\mathbf{B}}(m) \cup d^\varepsilon(\overline{\mathbf{B}}(m)), D_2 = \text{span } \overline{\mathbf{B}}(m) \cup \partial(\overline{\mathbf{B}}(m))$$

where m ranges over all values with t_{m-1} a type (B) or $(M-S)$ singular t -value.

The result will follow from two claims.

Claim 1: D_1 and D_2 are acyclic subcomplexes.

Claim 2: $D_1 = D_2$

since then the identity map on the quotient will be an isomorphism of complexes.

Claim 1 is easy to verify since the composition of either differential with the projection to $\overline{\mathbf{A}}(m)$ is upper triangular with respect to a proper choice of ordering on the bases.

For Claim 2, let $p_1 = \pi_A \circ d^\varepsilon$ and $p_2 = \pi_A \circ \partial$ where

$$\pi_A : \overline{\mathbf{A}}(m-1) \oplus \overline{\mathbf{B}}(m) \oplus \overline{\mathbf{A}}(m) \rightarrow \overline{\mathbf{A}}(m-1) \oplus \overline{\mathbf{A}}(m)$$

denotes the projection. Clearly

$$D_1 = \text{span}\{x_{m;i,j}^-, p_1 x_{m;i,j}^-\}, \quad D_2 = \text{span}\{x_{m;i,j}^-, p_2 x_{m;i,j}^-\}$$

To see that $D_1 \subset D_2$ observe that

Case (B). Assuming $\{i, j\} \cap \{k, k+1\} = \emptyset$,

$$\begin{aligned} p_1 x_{m;i,j}^- &= p_2 x_{m;i,j}^- + \eta^{s_m}(k, j) p_2 x_{m;i,k+1}^- \\ p_1 x_{m;k,j}^- &= p_2 x_{m;k,j}^- + \eta^{s_m}(k, j) p_2 x_{m;k,k+1}^- \\ p_1 x_{m;k+1,j}^- &= p_2 x_{m;k+1,j}^- \\ p_1 x_{m;i,k}^- &= p_2 x_{m;i,k}^- + \sum_l \eta^{s_m}(l, k+1) p_2 x_{m;i,l}^- \\ p_1 x_{m;i,k+1}^- &= p_2 x_{m;i,k+1}^- \\ p_1 x_{m;k,k+1}^- &= p_2 x_{m;k,k+1}^- \end{aligned}$$

Case (M-S). Assuming $j \neq l$

$$\begin{aligned} p_1 x_{m;i,j}^- &= p_2 x_{m;i,j}^- \\ p_1 x_{m;i,l}^- &= p_2 x_{m;i,l}^- + p_2 x_{m;i,k}^- \end{aligned}$$

Remark. It should be apparent at this point why we don't explicitly need to know the terms ${}^\circ x^-, {}^\circ y, {}^\circ z$ in the formulas for d^ε .

6 Additional remarks

6.1 A different approach to generating families and linearized contact homology

Our goal here is to provide a sketch of a proof of a statement similar, but not equivalent to Theorem 5.3. Here we assume to be given a front diagram L of a Legendrian knot possessing a graded normal ruling. We apply to this diagram the construction of Section 3, and get a generating family of functions $\{f_t : \mathbb{R}^N \rightarrow \mathbb{R}\}$ for L . As in Section 5, we consider the function $w : \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$, $w(x, y, t) = f_t(y) - f_t(x)$. We consider the manifold with boundary, $W = \{(x, y, t) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R} \mid w(x, y, t) \geq \delta\}$ where δ is

a sufficiently small positive number. We are going to relate the homology of $(W, \partial W)$ to the homology of the Chekanov-Eliashberg DGA of the splashed diagram linearized by the augmentation constructed in Section 4.2.

What makes the approach here different from the approach of Section 5, is that we consider the Morse complex of the function w (with respect to the standard metric in W), rather than the Morse complexes of individual functions f_t .

Proposition 5.2 creates a temptation to compare the Morse complex for the function w on W and the linearized DGA for an augmentation of the front diagram (we need to compare only differentials: the chain spaces are the same by Proposition 5.2). The difficulty is that in general we do not have any explicitly constructed augmentation. By this reason, we assume that our diagram L is a Ng's diagram modified by the splash construction (see Figure 7). (Remark, that neither the Ng construction nor the splashing will affect the topology of the pair $(W, \partial W)$, so our theorem concerns an arbitrary front diagram with a graded normal ruling.) Also, we will make some additional assumptions about the family f_t (which are easy to satisfy within the construction of Section 3). First, we assume that within laminated zones and inserts containing non-switching crossings, the critical points of the functions f_t do not depend on t . One more condition will be formulated later.

For the augmentation ε we take the augmentation constructed in Section 4.2.

Theorem 6.1 *Let $L, \{f_t\}, W, w$ and ε be as described above. Then, the Morse complex of the function w (with respect to the Euclidean metric in W) is isomorphic, up to a dimension shift by $N + 1$, to the linearized Chekanov-Eliashberg DGA for the augmentation ε as described in Proposition 4.3.*

Since the function w is constant on ∂W and has no critical points on ∂W , we have the following

Corollary 6.1 *The homology of $(W, \partial W)$ coincides with the homology of the Chekanov-Eliashberg DGA linearized by the augmentation ε with the shift of dimensions by $N + 1$.*

Proof of Theorem 6.2 Consider the critical point (x_0, y_0, t_0) of the function w corresponding, in the sense of Proposition 6.1, to the crossing $x_{m;ij}^-$ of the splashed diagram. Then the points $(t_0, f_{t_0}(x_0)), (t_0, f_{t_0}(y_0))$ belong to

the front diagram L , and the tangents to L at these two points are parallel (see Fig. 10).

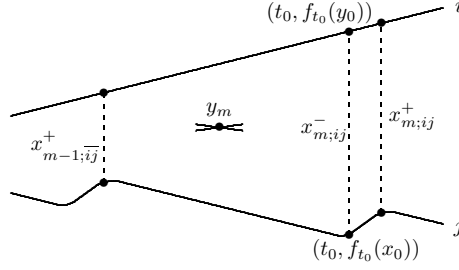


Figure 10: In the Morse complex, $dx_{m;ij}^-$ contains $x_{m;ij}^+$ and $x_{m-1;i\bar{j}}^+$

Let i, j be the numbers of these strands (in the m -th laminated zone). We assume that m -th insert contains a crossing y_m which is not a switch. The line (x_0, y_0, t) in $\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}$ is tangent to the gradient of w at the segment $J = [t_1, t_2] \ni t_0$ provided that x_0 and y_0 remain critical points of the function f_t for $t \in J$. The variation of the point (x_0, y_0, t_0) along this line corresponds to the simultaneous variations of the points $(t_0, f_{t_0}(x_0))$, $(t_0, f_{t_0}(y_0))$ along their strands of L , to the left or to the right, provided that the line of these two points remain vertical. Since in the both directions of the variations of t , the distance between these two points decreases (Fig. 10), (x_0, y_0, t) , $t \geq t_0$ and (x_0, y_0, t) , $t \leq t_0$ are descending gradient trajectories of the function w . These trajectories stop when we arrive at the diagram L , at the pair of points where the tangents to the strands again become parallel. If one of the strands, passes through y_m , then nothing changes (provided that y_m is not a switch). Thus, in these cases the differential of $x_{m;ij}^-$ in the Morse complex of w contains $x_{m;ij}^+$ and $x_{m-1;i\bar{j}}^+$.

If we move from the point (x_0, y_0, t_0) along some other descending gradient trajectory, then x_0 and y_0 , if move at all, move along the gradient trajectory of the function f_t ending at some other critical point of f_t (this trajectory is the same for all t : a vertical line). This give the following two additional possibilities for the terms in the differential of $x_{m;ij}^-$ in the Morse complex of w . If $i < \tau(i) < j$, then y_0 ascends to the companion critical point (with a smaller critical value), while x_0 stands unchanged. If $i < \tau(j) < j$, then x_0 descends to the companion critical point, while y_0 stands unchanged. These variations of the point (x_0, y_0, t_0) are shown on Figure 11.

We see now that in the case (ii) of Proposition 4.3, the differential d^ε of

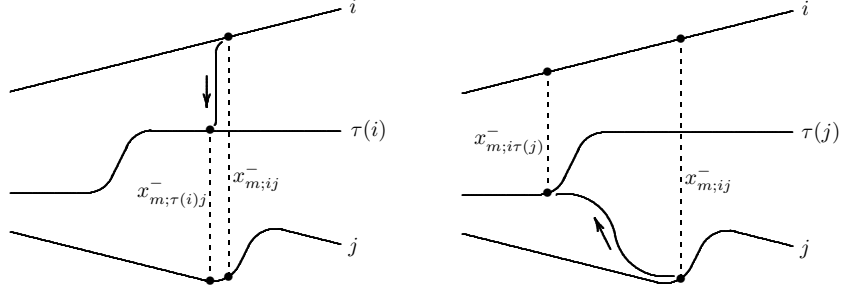


Figure 11: $dx_{m;ij}^-$ contains $x_{m;\tau(i)j}^-$, if $i < \tau(i) < j$ and $x_{m;i\tau(j)}^-$, if $i < \tau(j) < j$

the linearized complex of the Chekanov-Eliashberg DGA is the same as the differential of the Morse complex of the function w . In the same way, we can obtain the same result in the cases (i), (iv), and (v) of Proposition 4.3. In Case (i), we will not have anything similar to Fig. 10, and Fig. 11 should be modified: the arrows will be directed to x^+ rather than x^- (one should remember that we consider descending gradient trajectories joining critical points with the difference 1 between the indices). The difference between Cases (iv) and (v) and Case (ii) is negligible.

The remaining case (iii) (of a switch) requires some additional arguments. First, the critical points of the functions f_t are not fixed (with respect to t) any more. But we can assume that their movement is slow (compared with the gradients of f_t at a distance from the critical points). In this case, a gradient trajectory of the function w will not follow the strands of L , but will be close to them, and will return to these strands as soon as the critical points stabilize. This shows that all the summands of the differential of the Morse complex for w which we observed in the non-switch case, will stay in the switch case; but some new summands appear.

Let us begin from the case 2 of a switch: $\tau(k) < k$, $k+1 < \tau(k+1)$.

Figure 5, right, shows that, in addition to descending gradient trajectories from $\tau(k)$ to k and from $k+1$ to $\tau(k+1)$ (from A to b and from a to α , in the notations of Figure 5), there arise gradient trajectories from $\tau(k)$ to $k+1$ and from k to $\tau(k+1)$ (from A to a and from b to α) and this happens before (when we are moving from the right to the left) the crossing y_m (at least, on the xy diagram). Then the old trajectories disappear, and the new trajectories stay. Here comes a new condition on the slashes: the slashes on the trajectories numbers $\leq k+1$ should be placed between the moment

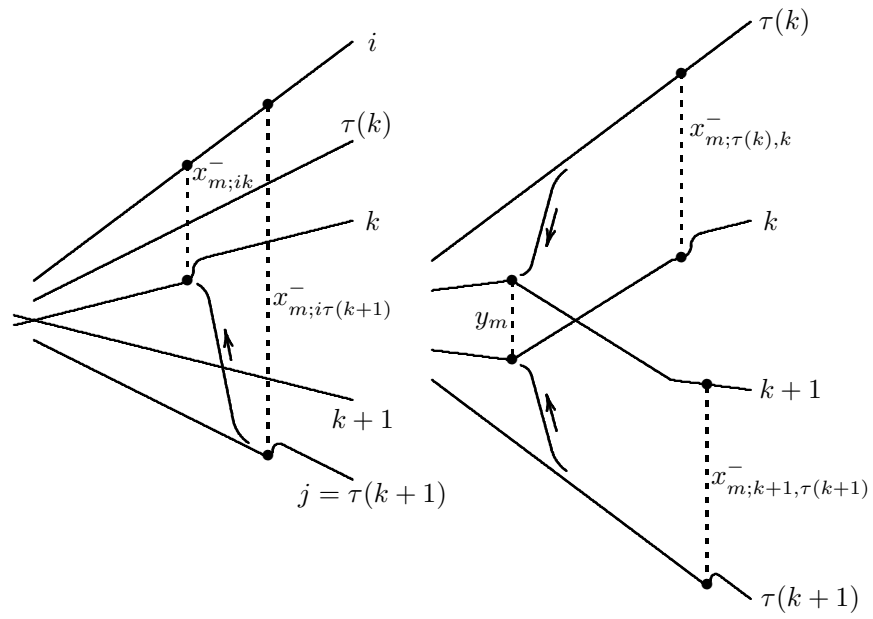


Figure 12: Case 2 of a switch: $dx_{m;i\tau(k+1)}^-$ involves $x_{m;ik}^-$ and $dx_{m;\tau(k),k}^-$ and $dx_{m;k+1,\tau(k+1)}^-$ involve y_m

of arising the new trajectories and the crossing. Figure 13, left, shows how the summand $x_{m;ik}^-$ arises in $dx_{m;i\tau(k+1)}^-$ (the strand i may be above or below the strand $\tau(k)$). Figure 12, right (where we have to present a more detailed picture of a neighborhood of a crossing) shows how the summand y_m arises in $dx_{m;\tau(k),k}^-$ and $dx_{m;k+1,\tau(k+1)}^-$. This settles the whole formula in Case 2.

Notice that the transition in Figure 12, left, does not involve the strand $\tau(k)$, and hence is also valid in Case 3 of a switch ($k+1 < \tau(k+1) < \tau(k)$); thus, in this case $dx_{m;i\tau(k+1)}^-$ involves $x_{m;ik}^-$. Similarly, according to Figure 13, right, $dx_{m;\tau(k),k}^-$ involves y_m in Case 1 ($\tau(k+1) < \tau(k) < k$) and $dx_{m;k+1,\tau(k+1)}^-$ involves y_m in Case 3 (we keep in these cases our assumption regarding the location of splashes).

For Case 1 of a switch, let us examine Figure 5, left. In the bottom diagram (corresponding to the maximal value of t), we see descending gradient trajectories $A \rightarrow b$ and $B \rightarrow a$. Then the first one becomes $A \rightarrow B \rightarrow b$, after which the (short-living) trajectory $A \rightarrow B$ disappears and $B \rightarrow b$ remains. Also, there appears a trajectory $A \rightarrow a$ and (after that) the trajectory $B \rightarrow a$ disappears (all these "after" mean "for a smaller value of t ").

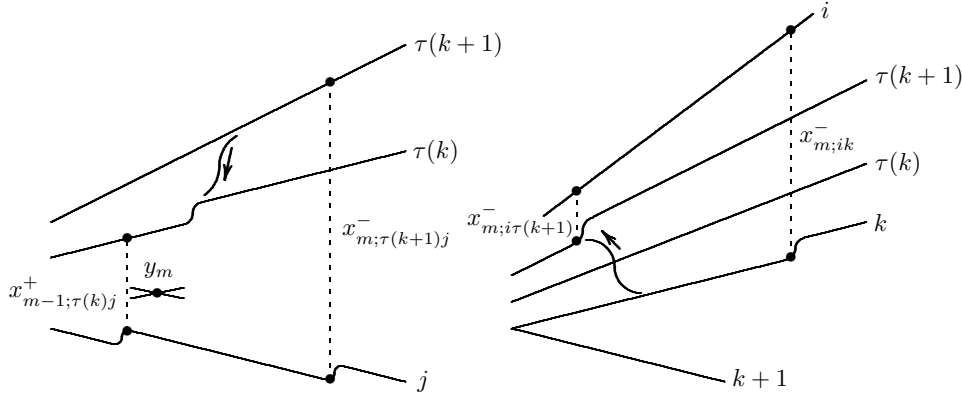


Figure 13: Case 1 of a switch: $dx_{m;\tau(k+1)j}^-$ involves $x_{m-1;\tau(k)j}^+$ and $dx_{m;ik}^-$ involves $x_{m;i\tau(k+1)}^-$

The corresponding descending gradient trajectories of the function w are shown on Figure 13 (and also on Figure 12, right, see the explanation above). Certainly, we also need a right arrangement of splashes: the splashes on the strands with the numbers $\tau(k+1)$ and below should be closer to the crossing y_m than all the trajectories described above. Also, the strand number j on

the diagram of Figure 13, left, may be above the crossing y_m , or even may contain it.

This settles Case 1 of the switch; Case 3 is similar.

6.2 The Sabloff duality is the Alexander duality

In [S1] Sabloff established a duality theorem for the linearized contact homology groups $H(\mathcal{A}^\varepsilon)$.

Theorem 6.2 (Sabloff) *If ℓ is a Legendrian knot and $\varepsilon : \mathcal{A} \rightarrow \mathbb{Z}_2$ any graded augmentation, then we have*

$$\begin{aligned} \dim_{\mathbb{Z}_2} H_k(\mathcal{A}^\varepsilon) &= \dim_{\mathbb{Z}_2} H_{-k}(\mathcal{A}^\varepsilon) & k \neq \pm 1 \\ \dim_{\mathbb{Z}_2} H_1(\mathcal{A}^\varepsilon) &= \dim_{\mathbb{Z}_2} H_{-1}(\mathcal{A}^\varepsilon) + 1. \end{aligned}$$

Together with our Theorem 5.3, this statement gives the following relation for the homology groups of the pair $(W_{\geq \delta}, W_\delta)$ with sufficiently small $\delta > 0$.

Theorem 6.3

- (i) *If $k \neq \pm 1$, then $H_{n+k+1}(W_{\geq \delta}, W_\delta; \mathbb{Z}_2) \cong H_{n-k+1}(W_{\geq \delta}, W_\delta; \mathbb{Z}_2)$.*
- (ii) *$H_{n+2}(W_{\geq \delta}, W_\delta; \mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus H_n(W_{\geq \delta}, W_\delta; \mathbb{Z}_2)$.*

In this section we provide a proof of Part (i) from the generating family perspective. Probably, Part (ii) can be proven by a careful examination of the homomorphisms in the long exact sequence used below.

Lemma 6.1 *In the region $\delta \geq w \geq -\delta$, w is a Morse-Bott function. There is a single non-degenerate critical submanifold diffeomorphic to S^1 and contained in the level $w = 0$.*

Proof. It is easy to see that for small enough δ the critical set will be $\Delta = \{(x, x, t) \in \mathbb{R}^{2n+1} \mid (x, t) \in S_F\}$. The Hessian matrix at $(x, x, t) \in \Delta$ has the block form

$$H_{(x,x,t)} = \begin{bmatrix} A & 0 & b \\ 0 & -A & -b \\ b^T & -b^T & 0 \end{bmatrix}$$

where $A = A^T = \left[\frac{\partial^2 F}{\partial x_i \partial x_j}(x, t) \right]$ and $b = \left[\frac{\partial^2 F}{\partial x_i \partial t}(x, t) \right]$. Under the transversality assumption on F , $[A \ b]$ is of full rank. To verify that Δ is non-degenerate we must check that $\ker H_{(x,x,t)} = T_{(x,x,t)}\Delta$. Since

$$S_F = \left\{ (x, t) \in \mathbb{R}^n \times \mathbb{R} \mid \frac{\partial F}{\partial x_i}(x, t) = 0, 1 \leq i \leq n \right\}, \quad T_{(x,t)}S_F = \text{Ker}[A \ b],$$

and hence

$$T_{(x,x,t)}\Delta = \{(\xi, \eta, \tau) \in T_{(x,x,t)}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}) \mid (\xi, \tau) \in \text{Ker}[A \ b]\}.$$

Now,

$$(\xi, \eta, \tau) \in \ker H_{(x,x,t)} \Leftrightarrow \begin{bmatrix} A\xi + \tau b \\ -A\eta - \tau b \\ b^T(\xi - \eta) \end{bmatrix} = 0 \quad 6.1$$

We see immediately that $T_{(x,x,t)}\Delta \subset \ker H_{(x,x,t)}$. For the reverse inclusion, suppose that $(\xi, \eta, \tau) \in \ker H_{(x,x,t)}$. From (6.1) we see that $(\xi, \tau), (\eta, \tau) \in \ker [A \ b]$. In addition, $\xi - \eta \in \ker [A \ b]^T = \{0\} \Rightarrow \xi = \eta$, so $T_{(x,x,t)}\Delta \supset \ker H_{(x,x,t)}$ holds.

For the index computation let $T_{(x,x,t)}\mathbb{R}^{2n+1} = h^+ \oplus h^- \oplus \ker H_{(x,x,t)}$ where the direct sum is orthogonal with respect to $H_{(x,x,t)}$ and the hessian is positive (resp. negative) definite when restricted to h^+ (resp. h^-). Such a decomposition exists for any symmetric bilinear form defined on a vector space over \mathbb{R} , and $\text{ind}(w, \Delta) = \dim h_-$ is well defined. Now, note that the isomorphism $S : T_{(x,x,t)}\mathbb{R}^{2n+1} \rightarrow T_{(x,x,t)}\mathbb{R}^{2n+1}$, $S(\xi, \eta, \tau) = (\eta, \xi, \tau)$ satisfies $H_{(x,x,t)}(Su, Sv) = -H_{(x,x,t)}(u, v)$ for any $u, v \in T_{(x,x,t)}\mathbb{R}^{2n+1}$. Therefore, $T_{(x,x,t)}\mathbb{R}^{2n+1} = S(h^-) \oplus S(h^-) \oplus \ker H_{(x,x,t)}$ is another $H_{(x,x,t)}$ -orthogonal direct sum, and now $S(h^-)$ is positive definite and $S(h^+)$ is negative definite. It follows that $\dim h^+ = \dim h^- \Rightarrow \text{ind}(w, \Delta) = n$.

Corollary 6.2 $H_j(W_{\leq \delta}, W_{\leq -\delta}; \mathbb{Z}_2) \cong H_{j-n}(S^1, \mathbb{Z}_2)$, hence

$$\dim H_j(W_{\leq \delta}, W_{\leq -\delta}; \mathbb{Z}_2) = \begin{cases} 1, & \text{if } j = n, n+1 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. It follows from Lemma 6.1 and a fundamental result from the Morse-Bott theory that $W_{\leq \delta}$ is homotopy equivalent to $W_{\leq -\delta}$ with the total space of a disk bundle $E_- \rightarrow \Delta$ of dimension $\text{ind}(w, \Delta)$ attached along ∂E_- . We have

$$H_j(W_{\leq \delta}, W_{\leq -\delta}; \mathbb{Z}_2) \cong H_j(E_-, \partial E_-; \mathbb{Z}_2) \cong H_{j-n}(S^1; \mathbb{Z}_2)$$

where the last \cong is the Thom isomorphism.

Lemma 6.2 *If $r \neq n-1, n, n+1$, then the inclusion homomorphism*

$$H_r(W_{\leq -\delta}; \mathbb{Z}_2) \rightarrow H_r(W_{\leq \delta}; \mathbb{Z}_2)$$

is an isomorphism.

Proof: Corollary 6.2 and the homological sequence of the pair $(W_{\leq -\delta}, W_{\leq \delta})$.

Lemma 6.3 *Let $V = \mathbb{R}^{N+1}$, $B \subset V$ be a ball, and $V_- \subset V$ be a half space. Suppose that V is covered by two $N+1$ -dimensional manifolds with boundary, $V = M_+ \cup M_-$, $\partial M_+ = \partial M_- = M_+ \cap M_-$. Suppose also that $M_- \cup (V - B) = V_- \cup (V - B)$. Then, for any q ,*

$$\tilde{H}_q(M_+; \mathbb{Z}_2) \cong \tilde{H}_{N-q}(M_-; \mathbb{Z}_2).$$

Proof: this is a version of the Alexander duality.

Proof of Theorem 6.3 (i) If $k = 0$, we have nothing to prove. Let $k \neq 0, \pm 1$.

$$\begin{aligned} & H_{n+k+1}(W_{\geq \delta}, W_{\delta}; \mathbb{Z}_2) \\ \cong & H_{n+k+1}(\mathbb{R}^{2n+1}, W_{\leq \delta}; \mathbb{Z}_2) \quad (\text{excision}) \\ \cong & \tilde{H}_{n+k}(W_{\leq \delta}; \mathbb{Z}_2) \quad (\text{homological sequence of } (\mathbb{R}^{2n+1}, W_{\leq \delta})) \\ \cong & \tilde{H}_{n+k}(W_{\geq -\delta}; \mathbb{Z}_2) \quad (\text{homeomorphism } (x, y, t) \mapsto (y, x, t)) \\ \cong & \tilde{H}_{n-k}(W_{\leq -\delta}; \mathbb{Z}_2) \quad (\text{Lemma 6.3}) \\ \cong & \tilde{H}_{n-k}(W_{\leq \delta}; \mathbb{Z}_2) \quad (\text{Lemma 6.2}) \\ \cong & H_{n-k+1}(\mathbb{R}^{2n+1}, W_{\leq \delta}; \mathbb{Z}_2) \quad (\text{homological sequence of } (\mathbb{R}^{2n+1}, W_{\leq \delta})) \\ \cong & H_{n-k+1}(W_{\geq \delta}, W_{\delta}; \mathbb{Z}_2) \quad (\text{excision}) \end{aligned}$$

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